$\mathcal{H}_{xy} = \frac{1}{2} (-\nabla^2 + \rho^2)$.

$\rho^2 = x^2 + y^2$ and $\hbar = m = \omega = 1$.

Griffith's Eq. 2.71:

$$H_{n+1}(\xi) = 2\xi\, H_n(\xi) - 2n\, H_{n-1}(\xi)$$

In 2D cartesian coordinates, the del operator is defined

$$\nabla f = [\partial/\partial x\, f,\ \partial/\partial y\, f].$$

$$\nabla^2 f = \nabla\cdot\nabla f = \nabla\cdot[\partial/\partial x\, f,\ \partial/\partial y\, f]$$
$$= \partial^2/\partial x^2\, f + \partial^2/\partial y^2\, f.$$

$\therefore\ \nabla^2 = \partial^2/\partial^2 x + \partial^2/\partial^2 y$.

Then,

$$\mathcal{H}_{xy} = \frac{1}{2} (-(\partial^2/\partial^2 x + \partial^2/\partial^2 y) + (x^2 + y^2)).$$

$$\mathcal{H}_{xy} = \frac{1}{2} (-(\partial^2/\partial^2 x + \partial^2/\partial^2 y) + (x^2 + y^2))$$
$$= \frac{1}{2} (x^2 - \partial^2/\partial^2 x + y^2 - \partial^2/\partial^2 y)$$
$$= \frac{1}{2}(x^2 - \partial^2/\partial^2 x) + \frac{1}{2}(y^2 - \partial^2/\partial^2 y).$$

$$\mathcal{H}_x + \mathcal{H}_y = \frac{1}{2}(x^2 - \partial^2/\partial^2 x) + \frac{1}{2}(y^2 - \partial^2/\partial^2 y).$$

∎

The Schrodinger Equation then reads,

$$[\tfrac{1}{2}(x^2 - \partial^2/\partial^2 x) + \tfrac{1}{2}(y^2 - \partial^2/\partial^2 y)]\ \Psi = (E_x + E_y)\ \Psi.$$

Assuming a separable solution $\Psi(x,y) = X(x)\ Y(y)$, with $E = E_x + E_y$.

$$[\tfrac{1}{2}(x^2 - \partial^2/\partial^2 x) + \tfrac{1}{2}(y^2 - \partial^2/\partial^2 y)]\ X(x)\ Y(y) = (E_x + E_y)\ X(x)\ Y(y).$$

$$\tfrac{1}{2}(x^2 - \partial^2/\partial^2 x)\ X(x)\ Y(y) + \tfrac{1}{2}(y^2 - \partial^2/\partial^2 y)\ X(x)\ Y(y)$$
$$= (E_x + E_y)\ X(x)\ Y(y).$$

$$[1/X(x)]\ [\tfrac{1}{2}(x^2 - \partial^2/\partial^2 x)\ X(x)] + [1/Y(y)]\ [\tfrac{1}{2}(y^2 - \partial^2/\partial^2 y)\ Y(y)]$$
$$= (E_x + E_y).$$

So, I have two differential equations,

$$\tfrac{1}{2}(x^2 - \partial^2/\partial^2 x)\ X(x) = E_x X(x), \text{ and}$$
$$\tfrac{1}{2}(y^2 - \partial^2/\partial^2 y)\ Y(y) = E_y Y(y).$$

The solutions to these differential equations are the same as for the 1D harmonic oscillator. They have eigenvalues $(n + 1/2)$, where $\hbar = \omega = 1$, with $n = 0,1,2,...$.

$\therefore$ the eigenvalues for the combined operator are $n_x + n_y + 1$.

The degeneracy is pretty obvious, just from counting the possibilities: there is n+1 degeneracy for each value of $n = n_x + n_y$.

So,

n	degeneracy
0	1
1	2
2	3
3	4
4	5
5	6

The Hermite polynomials help to generate the eigenstates of this:

$H_n(x) = (-1)^n \exp(x^2)\ d/dx^n \exp(-x^2/2) = (2x - d/dx)^n * 1.$

The first six polynomials are

$H_0(x) = 1.$
$H_1(x) = 2x.$
$H_2(x) = 4x^2 - 2.$
$H_3(x) = 8x^3 - 12x.$
$H_4(x) = 16x^4 - 48x^2 + 12.$
$H_5(x) = 32x^5 - 160x^3 + 120x.$
$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120.$

The wave functions involving these polynomials, with the unitizations given in the intro, are

$$\Psi_{nm}(x) = \pi^{\wedge}(-1/4)\ 1/\sqrt{(2^n\ n!)}\ H_n(x) \exp(-x^2/2)\ \pi^{\wedge}(-1/4)\ 1/\sqrt{(2^m\ m!)}\ H_m(y) \exp(-y^2/2).$$

$$\Psi_{nm}(x) = 1/\sqrt{\pi}\ 1/\sqrt{(2^n\ n!\ 2^m\ m!)}\ H_n(x)\ H_m(y) \exp(-(x^2/2 + y^2/2)).$$

The lowering operators are

$a_x = 1/\sqrt{2}\ (x + \iota p_x)$ and
$a_y = 1/\sqrt{2}\ (y + \iota p_y).$

They are not hermitian, but x,y and p_x, p_y are, so the raising operators are

$a_x^+ = 1/\sqrt{2}\ (x - \iota p_x)$ and
$a_y^+ = 1/\sqrt{2}\ (y - \iota p_y).$

Applying these to the ground state $|00\rangle$, I can find the first six states, with normalization:

$$a_x^+|00\rangle = 1/\sqrt{2}\ (x - \iota p_x)|00\rangle = 1/\sqrt{2}\ (x|00\rangle - \iota p_x|00\rangle)$$

b) & c)

I'm still working out the algebra, here. I will try to finish it as soon as I can, but I know I also have new work to do.

I finished much of this assignment, but need to get done faster in the future.