**University of Kentucky, Physics 521**
**Homework #12, Rev. A, due Wednesday, 2018-01-17**

**0.** Griffiths [2ed] Ch. 3 #7, #10, #13, #14, #15; Ch. 4 #18, #19.

**1.** Consider a two-dimensional isotropic harmonic oscillator with Hamiltonian $\mathcal{H}_{xy} = \frac{1}{2}\left(-\nabla^2 + \rho^2\right)$, where $\rho^2 = x^2 + y^2$, and $\hbar = m = \omega = 1$ so that $\rho$ plays the role of the normalized coordinate $\xi$ of Griffiths Eq. [2.71].

**a)** Show that the Hamiltonian separates into two independent oscillators $\mathcal{H}_{xy} = \mathcal{H}_x + \mathcal{H}_y$ in cartesian coordinates, and thus the energy levels are $E_{n_x n_y} = n_x + n_y + 1$. Identify the degeneracy of each energy level. Write the wave functions of the lowest six levels in terms of both Hermite polynomials $H_{n_x}(x)H_{n_y}(y)$ and creation operators $a^\dagger_{x,y}$ acting on the ground state $|n_x n_y\rangle = |00\rangle$. The two annihilation operators $a_x = \frac{1}{\sqrt{2}}(x + ip_x)$ and $a_y = \frac{1}{\sqrt{2}}(y + ip_y)$ act in independent directions.

**b)** The eigenfunctions of part a) do not have definite angular momentum. To obtain eigenstates of definite $L_z$ using the operator method, define annihilation operators for right and left circular quanta $a_r = \frac{1}{\sqrt{2}}(a_x - ia_y)$ and $a_\ell = \frac{1}{\sqrt{2}}(a_x + ia_y)$, respectively. Show that the only nonzero commutators between $a_r, a_r^\dagger, a_\ell, a_\ell^\dagger$ are $[a_r, a_r^\dagger] = [a_\ell, a_\ell^\dagger] = 1$. Show that $\mathcal{H}_{xy} = N_x + N_y + 1 = N_r + N_\ell + 1$ and $L_z = i(a_x a_y^\dagger - a_x^\dagger a_y) = N_r - N_\ell$, where $N_i = a_i^\dagger a_i$ as usual for $i = x, y, r, \ell$. Show also that $[\mathcal{H}_{xy}, a_i^\dagger] = a_i^\dagger$, so that $a_x, a_y$ and $a_r, a_\ell$ both act as ladder operators for two independent sets of quanta $|n_x n_y\rangle$ and $|n_r n_\ell\rangle$. From this we infer that $E_{nm} = n + 1$, and $L_z = m$, where $n = n_r + n_\ell$ and $m = n_r - n_\ell$. Plot the energy levels $E_{nm}$ versus $m$ and show that the allowed values of $n$ are $2k + |m|$ where $k = 0, 1, 2\ldots$ and $m = 0, \pm 1, \pm 2\ldots$. Note the checkerboard pattern with every other value of $n$ missing.

**c)** Use the coordinate representation of $a^\dagger_{r,\ell}$ acting on the ground state $|n_r n_\ell\rangle = |00\rangle$ to obtain wavefunctions for the six $n \le 2$ states and compare your results with H11 part e). Expand the $a^\dagger_{r,l}$ in terms of $a^\dagger_{x,y}$ and simplify to show that the $|n_l, n_r\rangle$ states of part b) are linear combinations of the $|n_x, n_y\rangle$ states of part a).