# RECURSION AND GENERATING FUNCTIONS

## 1. Recursive Sequence: Technical Aspect

1.1. **General information.** A recursive sequence is a sequence $\{a_n : n \geq 0\}$ of numbers satisfying

$$a_n = f(n, a_{n-1}, a_{n-2}, ..., a_{n-k}, ...) \tag{1.1}$$

for any $n$, i.e., the $n$-th entry of the sequence is uniquely determined by the entries before it. And (1.1) is called a recursive relation.

Look at (1.1) and one realizes that it is very close to a differential equation. Indeed, a recursive sequence is a discrete version of a differential equation.

Recursive sequences are also closely related to generating functions, as we will see.

To study a recusive sequence $\{a_n\}$, one would like to have a closed formula for $a_n$. However, a closed formula is only available in the most ideal situation, much like differential equations. For example, we know how to find the formula for a linear recursion.

1.2. **Linear recursion.** A linear recursion is a recursive sequence satisfying

$$a_n = p_1 a_{n-1} + p_2 a_{n-2} + ... + p_k a_{n-k} \tag{1.2}$$

where $p_1, p_2, ..., p_k$ are constants.

The most famous example is the Fibonacci sequence $a_n = a_{n-1} + a_{n-2}$ with $a_0, a_1$ given.

The standard method to derive the closed formula for linear recursion is via generating function. Let $\{a_n\}$ be the Fibonacci sequence. We let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $f(x)$ encodes all the information about $\{a_n\}$.

Note that $xf(x) = \sum_{n=1}^{\infty} a_{n-1} x^n$ and $x^2 f(x) = \sum_{n=2}^{\infty} a_{n-2} x^n$. So

$$\begin{aligned} xf(x) + x^2 f(x) &= a_0 x + \sum_{n=2}^{\infty} (a_{n-2} + a_{n-1}) x^n \\ &= a_0 x + \sum_{n=2}^{\infty} a_n x^n = a_0 x + f(x) - a_0 - a_1 x \end{aligned} \tag{1.3}$$

In the end, we are solving the equation

$$xf(x) + x^2 f(x) = f(x) - a_0 - (a_1 - a_0)x \tag{1.4}$$

with $f(x)$ as the unknown function. The solution is

$$f(x) = \frac{a_0 + (a_1 - a_0)x}{1 - x - x^2} \tag{1.5}$$

To get the formula for $a_n$, we have to expand the Taylor series for $f(x)$ obtained above. Note that $f(x)$ is a rational function. In order to expand $f(x)$, we write $f(x)$ as the sum of partial fractions:

$$\frac{a_0 + (a_1 - a_0)x}{1 - x - x^2} = \frac{a_0 + (a_1 - a_0)x}{(r_1 - x)(r_2 - x)} = \frac{c_1}{r_1 - x} + \frac{c_2}{r_2 - x} \tag{1.6}$$

where $r_1, r_2$ are the two roots of $1 - x - x^2 = 0$ and $c_1, c_2$ depend on the values of $a_0, a_1$. And the RHS of (1.6) can be expanded into Taylor series:

$$\begin{aligned} \frac{c_1}{r_1 - x} + \frac{c_2}{r_2 - x} &= c_1 r_1^{-1}(1 - r_1^{-1}x)^{-1} + c_2 r_2^{-1}(1 - r_2^{-1}x)^{-1} \\ &= c_1 r_1^{-1} \sum_{n=0}^{\infty} r_1^{-n} x^n + c_2 r_2^{-1} \sum_{n=0}^{\infty} r_2^{-n} x^n \\ &= \sum_{n=0}^{\infty} (c_1 r_1^{-n-1} + c_2 r_2^{-n-1}) x^n \end{aligned} \tag{1.7}$$

So $a_n = c_1 r_1^{-n-1} + c_2 r_2^{-n-1}$. We may rewrite the solution as

$$a_n = C_1 r_1^{-n} + C_2 r_2^{-n} \tag{1.8}$$

where $C_1, C_2$ are constants determined by $a_0, a_1$ (think of the values $a_0, a_1$ as initial conditions).

Of course, to carry the real computation is not very fun. But I suggest that everyone should do it at least once in their lifetime.

*Exercise* 1.1. Let $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Find a formula for $a_n$.

In some situations, we do not really need (1.8) and one finds that (1.5) is more useful.

*Exercise* 1.2. Let $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Find $a_0 + \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \ldots$.

*Exercise* 1.3. Let $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Use or not use the formula for $a_n$ to show the following: for every prime number $p$, there exists $a_n$ such that $p | a_n$. Try to do it in both ways.

In general, we can solve a linear recursion (1.2) in much the same way. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then we have the functional equation:

$$p_1 x f(x) + p_2 x^2 f(x) + \ldots + p_k x^k f(x) = f(x) - G(x) \tag{1.9}$$

where $G(x) = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + \ldots + (a_{k-1} - a_{k-2} - \ldots - a_0)x^{k-1}$. So

$$f(x) = \frac{G(x)}{1 - p_1 x - p_2 x^2 - \ldots - p_k x^k}. \tag{1.10}$$

Next we write $f(x)$ as a sum of partial fractions and expand their Taylor series. At least, we know how to do it in theory. In the end, the formula looks like

$$a_n = C_1 r_1^{-n} + C_2 r_2^{-n} + ... + C_k r_k^{-n} \tag{1.11}$$

where $C_i$ are constants determined uniquely by $a_0, a_1, ..., a_{k-1}$ (initial conditions) and $r_i$ are the roots of

$$1 - p_1 x - p_2 x^2 - ... - p_k x^k = 0. \tag{1.12}$$

Here we assume that the above equation does not have multiple roots.

*Exercise* 1.4. What happens if (1.12) does have multiple roots?

There are several other ways to derive the closed formula for a linear recursion. Here is another one using a little linear algebra.

We work with the Fibonacci sequence $a_n = a_{n-1} + a_{n-2}$. Let $b_n = a_{n-1}$. Then

$$\begin{cases} a_n = a_{n-1} + b_{n-1} \\ b_n = a_{n-1} \end{cases} \tag{1.13}$$

Using matrix notation, we have

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} \tag{1.14}$$

Therefore,

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \tag{1.15}$$

Now the question is how to give a closed formula for $A^n$ given a matrix $A$. Here we need a little linear algebra: we will diagonalize $A$, i.e., find matrix $P$ such that $A = P^{-1}BP$ for some diagonal matrix $B$. Then $A^n = P^{-1}B^nP$. Again the real computation is messy. Diagonalizing a $3 \times 3$ matrix is no trivial matter by hand. But at least we know how to do it in theory. Or by any chance that you need to find $A^n$ for some large $A$, you can use computer (any linear algebra package has the routine of diagonalizing a matrix).

Actually (1.13) is a system of two linear recursions involving two sequences $\{a_n\}$ and $\{b_n\}$. The way we convert $a_n = a_{n-1} + a_{n-2}$ into (1.13) is very similar to the way we convert a second order differential equation into a system of two first order differential equations. Of course, this can be carried out in general.

1.3. **Linear recursion: nonhomogeneous case.** A nonhomogeneous linear recursion is a sequence satisfying

$$a_n = p_1 a_{n-1} + p_2 a_{n-2} + ... + p_k a_{n-k} + g_n \tag{1.16}$$

with the nonhomogeneous term $g_n$.

*Example* 1.5. Let $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2} + 1$ for $n \geq 2$. Find a formula for $a_n$.

This is Fabonacci sequence with a nonhomogeneous term. The generating function approach still works. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$xf(x) + x^2 f(x) + \sum_{n=2}^{\infty} x^n = f(x) - 1 \tag{1.17}$$

where $\sum_{n=2}^{\infty} x^n = x^2/(1-x)$ comes from the nonhomogeneous term 1. So

$$f(x) = \frac{1 - x + x^2}{(1-x)(1-x-x^2)} \tag{1.18}$$

Again we can find the answer using partial fractions.

Another way to do it is the following. Without the nonhomogeneous term, the general solution is

$$a_n = C_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + C_2 \left( \frac{1-\sqrt{5}}{2} \right)^n \tag{1.19}$$

And $a_n = a_{n-1} + a_{n-2} + 1$ has an obvious solution with $a_n = c$ constant. Of course, $c$ is the solution of $c = c + c + 1$ and $c = -1$. So the general solution for $a_n = a_{n-1} + a_{n-2} + 1$ is

$$a_n = C_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + C_2 \left( \frac{1-\sqrt{5}}{2} \right)^n - 1 \tag{1.20}$$

where $C_1$ and $C_2$ can be determined by $a_0 = a_1 = 1$. Please note the similarity between this and the way to solve a nonhomogeneous linear differential equation.

1.4. **Some other cases.** Other than linear recursions, there are very few other classes of recursions where a closed formula can be found. I will give some other examples. These examples are quite sporadic and do not have much pattern to them.

*Example* 1.6. Let $a_0 = 1$ and $a_n = na_{n-1} + 1$ for $n \geq 1$. Find a formula for $a_n$.

Let $a_n = (n!)b_n$. Then $b_n = b_{n-1} + 1/n!$. So

$$b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} \tag{1.21}$$

So

$$a_n = (n!) \left( 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} \right) \tag{1.22}$$

*Example* 1.7. Let $a_0 = 7$ and $a_n = 7 + \frac{1}{a_{n-1}}$. Find a formula for $a_n$.

This is actually the continuous fraction:

$$7 + \cfrac{1}{7 + \cfrac{1}{7 + ...}} \tag{1.23}$$

So naturally we write $a_n = p_n/q_n$. Then

$$\frac{p_n}{q_n} = \frac{7p_{n-1} + q_{n-1}}{p_{n-1}} \tag{1.24}$$

So

$$\begin{cases} p_n = 7p_{n-1} + q_{n-1} \\ q_n = p_{n-1} \end{cases} \tag{1.25}$$

Or equivalently, $p_n = 7p_{n-1} + p_{n-2}$. I would let you work out the rest.

Of course, if we want to find the value of (1.23), which is $\lim_{n\to\infty} a_n$, we do not really need the formula for $a_n$. Suppose that $\lim_{n\to\infty} a_n = x$ exists. Then

$$\lim_{n\to\infty} a_n = 7 + \lim_{n\to\infty} \frac{1}{a_{n-1}} \Rightarrow x = 7 + \frac{1}{x} \tag{1.26}$$

Solve it and we obtain $x = (7 + \sqrt{53})/2$. But with a formula for $a_n$, we can say much more about how fast $a_n$ converges to $x$.

*Exercise* 1.8. Show that there exists a constant $C$ such that

$$\left| \frac{p_n}{q_n} - \frac{7 + \sqrt{53}}{2} \right| \le \frac{C}{q_n^2} \tag{1.27}$$

for all $n$. Find the best possible $C$.

So in some sense, continuous fraction is the best possible way to approximate an irrational number by rational numbers. It is a little off topic here but I want to mention that the approximation (1.27) is also the best we can do if the irrational number concerned is algebraic, i.e., if $x$ is an algebraic number and $C, \varepsilon > 0$, then there are only finitely many pairs of integers $p, q$ such that

$$\left| \frac{p}{q} - x \right| \le \frac{C}{q^{2+\varepsilon}} \tag{1.28}$$

Of course, this is the famous Roth's theorem. Roth's original proof, though technical, is actually quite elementary, which is understandable by anyone with a decent background in analysis.

Now let us go back to our topic. In general, we can use the same method to work out the recursion

$$a_n = \frac{Aa_{n-1} + B}{Ca_{n-1} + D} \tag{1.29}$$

We let $a_n = p_n/q_n$. Then

$$\frac{p_n}{q_n} = \frac{Ap_{n-1} + Bq_{n-1}}{Cp_{n-1} + Dq_{n-1}} \tag{1.30}$$

We may let

$$\begin{cases} p_n = Ap_{n-1} + Bq_{n-1} \\ q_n = Cp_{n-1} + Dq_{n-1} \end{cases} \tag{1.31}$$

So

$$\begin{bmatrix} p_n \\ q_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \begin{bmatrix} a_0 \\ 1 \end{bmatrix} \tag{1.32}$$

This approach is very natural if you know something about projective geometry. The expression $(Az+B)/(Cz+D)$ actually gives an automorphism of the projective line $\mathbb{P}^1$. So it is natural to use the projective coordinates $z = p/q$.

*Example* 1.9. Let $a_0 = 1/4$ and $a_n = 2a_{n-1}^2 - 1$. Find a formula for $a_n$.

This is one example that if you have seen it before, you will most likely get it; if you have not, it takes quite a bit of luck to get it.

If we let $a_0 = \cos\theta$, then $a_1 = 2\cos^2\theta - 1 = \cos(2\theta)$, $a_2 = \cos(4\theta)$ and so on. We may take $\theta = \cos^{-1}(1/4)$. So $a_n = \cos(2^n\theta) = \cos(2^n \cos^{-1}(1/4))$.

## 2. Recursion: Applications

*Example* 2.1 (Putnam Problem Revisit). You have coins $C_1, C_2, ..., C_n$. For each $k$, $C_k$ is biased so that, when tossed, it has probability $1/(2k+1)$ of fallings heads. If the $n$ coins are tossed, what is the probability that the number of heads is odd? Express the answers as a rational function of $n$.

Terry worked out this problem using generating functions. I will do it using recursion.

Let $p_n$ be the probability that the number of heads is odd with $n$ coins are tossed. Then $1 - p_n$ is the probability that the number of heads is even. The recursion is

$$p_n = \left(1 - \frac{1}{2n+1}\right) p_{n-1} + \frac{1}{2n+1}(1 - p_{n-1}) = \frac{2n-1}{2n+1} p_{n-1} + \frac{1}{2n+1} \tag{2.1}$$

If we let $a_n = (2n+1)p_n$, then $a_n = a_{n-1} + 1$. So $a_n = n$ and $p_n = n/(2n+1)$.

*Example* 2.2 (Multiply of Immortal Rabbits). Suppose that each pair of rabbits mature in two months and gives birth to a pair of rabbits each month from then on. Assume that rabbits never die and we start with one pair of rabbits (new born). Find the number of pairs of rabbits after $n$ months.

This is the original problem where the Fabonacci sequence comes from. Let $a_n$ be the number of rabbits after $n$ months. Then $a_n = a_{n-1} + a_{n-2}$ with $a_0 = a_1 = 1$.

*Example* 2.3 (Putnam problem A-3 1999). Consider the power series

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n \tag{2.2}$$

Prove that for each integer $n \geq 0$, there is an integer $m$ such that

$$a_n^2 + a_{n+1}^2 = a_m \tag{2.3}$$

Although no recursive sequences are directly involved, it is worthwhile of noting that $a_n = 2a_{n-1} + a_{n-2}$. Of course, the way to find $a_n$ is using partial fractions as mentioned. Once $a_n$ is known, the rest is easy. I will let you work out the rest.

*Example* 2.4 (Putnam problem A-6 1999). The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}} \tag{2.4}$$

Show that, for all $n$, $a_n$ is an integer multiple of $n$.

We rewrite (2.4) as

$$\frac{a_n}{a_n - 1} = 6\left(\frac{a_{n-1}}{a_{n-2}}\right) - 8\left(\frac{a_{n-2}}{a_{n-3}}\right) \tag{2.5}$$

If we let $b_n = a_n/a_{n-1}$, then

$$b_n = 6b_{n-1} - 8b_{n-2} \tag{2.6}$$

Of course, we know how to solve (2.6). Once $b_n$ is known, we know $a_n = a_1(b_2b_3...b_n)$. I have not thought about it carefully. But it seems that you also need to know Fermat's little theorem to finish the proof. Anyway, I will leave the rest to you.

There are certainly zillions of examples concerning recursion. It is a recurring theme in Putnam (they love recursion, linear recursion in particular). If you are interested in the subject, you may pick up a standard textbook in combinatorics where you may find many interesting examples and applications. Probably, one of Richard Stanley's books is where to start.