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~~Otto~~ 4.0 1.1

4.28) From the most general normalized spinor χ, compute $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$, $\langle S_x^2 \rangle$, $\langle S_y^2 \rangle$, and $\langle S_z^2 \rangle$. Check that $\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \langle S^2 \rangle$.

Eq. 4.139: $\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$

In general, a and b must be normalized:

$|a|^2 + |b|^2 = 1.$

$\langle S_x \rangle = \ldots$

Recall $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ with

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\langle S_x \rangle = \frac{\hbar}{2}(a^* \; b^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(a^*b + b^*a)$$

$$\langle S_x \rangle = \hbar \, \mathrm{Re}(ab^*).$$

$$\langle S_y \rangle = \frac{\hbar}{2}(a^* \; b^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} i (b^*a - a^*b)$$

$$\langle S_y \rangle = -\hbar \, \mathrm{Im}(ab^*).$$

$$\langle S_z \rangle = \frac{\hbar}{2}(a^* \; b^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(a^*a - b^*b)$$

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4.28)

$$\langle S_z \rangle = \frac{\hbar}{2}\left(|a|^2 - |b|^2\right).$$

$$\langle S_z^2 \rangle = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{\hbar^2}{4}.$$

$$\langle S_x^2 \rangle = \frac{\hbar^2}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{\hbar^2}{4}.$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \frac{\hbar^2}{4}.$$

$$\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle \boxed{= \frac{3\hbar}{4} = \langle S^2 \rangle}.$$

4.58) Deduce the condition for minimum uncertainty in S_x and S_y (that is, equality in expression $\sigma_{S_x}\sigma_{S_y} \geq (\frac{\hbar}{2})|\langle S_z\rangle|$, for a particle of spin $\frac{1}{2}$ in the generic state.

Answer: With no loss of generality we can pick a to be real; then the condition for minimal uncertainty is that b is either pure real or else pure imaginary.

$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$. a and b normalized: $|a|^2+|b|^2=1$.

From problem 28,

$\langle S_x\rangle = \hbar\, Re\,(ab^*)$,

$\langle S_y\rangle = -\hbar\, Im\,(ab^*)$, and

$\langle S_z\rangle = \frac{\hbar}{2}(|a|^2-|b|^2)$.

$\sigma_{S_x}^2$ and $\sigma_{S_y}^2$ are the dispersion relationships

$\sigma_{S_x}^2 = \langle S_x^2\rangle - \langle S_x\rangle^2$ and

$\sigma_{S_y}^2 = \langle S_y^2\rangle - \langle S_y\rangle^2$.

$\sigma_{S_x}^2 = \frac{\hbar^2}{4} - (\hbar\, Re(ab^*))^2$.

(prob. 28)

$\sigma_{S_y}^2 = \frac{\hbar^2}{4} + (\hbar\, Im(ab^*))^2$.

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In polar form, $a = |a|e^{i\phi}$ and $b = |b|e^{i\phi'}$.

The phase difference is then $\phi - \phi'$.

$(\hbar \, Re(ab^*))^2 = (\hbar \, Re(|a|e^{i\phi}|b|e^{-i\phi'}))^2 \ldots$

~~set~~

Set $\theta = \phi - \phi'$,

$$\sigma_{S_x}^2 = \frac{\hbar^2}{4} - (\hbar \, Re(|a||b|e^{i\theta}))^2 = \frac{\hbar^2}{4} - \hbar^2 |a|^2|b|^2 \cos^2\theta.$$

$$\sigma_{S_y}^2 = \frac{\hbar^2}{4} - (\hbar \, Im(|a||b|e^{i\theta}))^2 = \frac{\hbar^2}{4} - \hbar^2 |a|^2|b|^2 \sin^2\theta.$$

$\sigma_{S_y}^2 \sigma_{S_x}^2 = \frac{\hbar^2}{4} \langle S_z \rangle^2$ when the uncertainty is at a minimum.

$$\left(\frac{\hbar^2}{4} - \hbar^2|a|^2|b|^2\cos^2\theta\right)\left(\frac{\hbar^2}{4} - \hbar^2|a|^2|b|^2\sin^2\theta\right)$$

$$= \left(\frac{\hbar^2}{4}\right)^2 \left[(1 - 4|a|^2|b|^2\cos^2\theta)(1 - 4|a|^2|b|^2\sin^2\theta)\right]$$

$$= 1 - 4|a|^2|b|^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + 16|a|^4|b|^4\cos^2\theta\sin^2\theta.$$

$$\frac{\hbar^2}{4}\langle S_z\rangle^2 = \frac{\hbar^2}{4}\frac{\hbar^2}{4}(|a|^2 - |b|^2)^2 = \left(\frac{\hbar^2}{4}\right)^2(|a|^4 - 2|a|^2|b|^2 + |b|^4).$$

$$1 - 4|a|^2|b|^2 + 16|a|^4|b|^4\cos^2\theta\sin^2\theta = |a|^4 - 2|a|^2|b|^2 + |b|^4.$$

$$1 + 16|a|^4|b|^4\cos^2\theta\sin^2\theta = \underbrace{|a|^4 + 2|a|^2|b|^2 + |b|^4}_{= (|a|^2 + |b|^2)^2 = 1}.$$

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$$1+16|a|^4|b|^4\cos^2\theta\sin^2\theta=1$$

$$16|a|^4|b|^4\cos^2\theta\sin^2\theta=0.$$

$$4|a|^2|b|^2\cos\theta\sin\theta=0.$$

$$|a|^2|b|^2\cos\theta\sin\theta=0.$$

Four solutions in the principle branch,

$\theta=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

So, either the phase difference must be one of these values to have minimum uncertainty between the two values. This is consistent with the conditions that if a is real, b must be totally real or totally imaginary.

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4.59 In classical electrodynamics the force on a particle of charge q moving with velocity v through electric and magnetic fields $\vec{E}$ and $\vec{B}$ is given by the Lorentz force law

$$F = q(\vec{E} + \vec{v} \times \vec{B}).$$

The force cannot be expressed as the gradient of a scalar potential energy function, and therefore the Schrödinger equation in its original form cannot accomodate it. The more sophisticated form,

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi,$$

has no problem. The classical Hamiltonian is

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi$$

where $\vec{A}$ is the vector potential $\vec{B} = \vec{\nabla} \times \vec{A}$ and φ the scalar potential $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$, so the Schrödinger equation becomes

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2 + q\varphi\right]\Psi$$

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a) show that

$$\frac{d\vec{r}}{dt} = \frac{1}{m}\langle(\vec{p} - q\vec{A})\rangle.$$

We know $\frac{d\langle\vec{r}\rangle}{dt} = \frac{i}{\hbar}\langle[H, \vec{r}]\rangle$ (eq. 3.71)
w/ $\langle\frac{\partial\langle\vec{r}\rangle}{\partial t}\rangle = 0$

$$[\hat{H}, \hat{\vec{r}}] = [\frac{1}{2m}(\vec{p}^2 - q(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}) + q^2A^2) + q\phi, \hat{\vec{r}}]$$

$$[\hat{H}, \hat{x}] = \frac{1}{2m}[p^2, x] - \frac{q}{2m}[(\vec{P}\cdot\vec{A} + \vec{A}\cdot\vec{P}), x].$$

(the commuting terms disappear)

$$[p^2, \hat{x}] = p_x[p_x, x] + [p_x, x]p_x = -i\hbar p_x - i\hbar p_x = -i2\hbar p_x.$$

$$[\vec{p}\cdot\vec{A}, \hat{x}] = [p_x A_x, x] = p_x[A_x, x] + [p_x, x]A_x = -i\hbar A_x.$$

$$[\vec{A}\cdot\vec{p}, \hat{x}] = [A_x p_x, x] = A_x[p_x, x] + [A_x, x]p_x = -i\hbar A_x.$$

$$[\hat{H}, \hat{x}] = \frac{1}{2m}(-i2\hbar p_x) - \frac{q}{2m}(-2i\hbar A_x) = -\frac{i\hbar}{m}(p_x - qA_x).$$

The same argument holds for $[\hat{H}, \hat{y}]$ and $[\hat{H}, \hat{z}]$:

$$[\hat{H}, \hat{\vec{r}}] = -\frac{i\hbar}{m}(\vec{p} - q\vec{A}).$$

$$\therefore \frac{d\langle\vec{r}\rangle}{dt} = \boxed{\frac{1}{m}\langle\vec{p} - q\vec{A}\rangle.}$$

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4.59 b) as always we identity $\frac{d\langle \vec{r}\rangle}{dt}$ with $\langle \vec{v}\rangle$.

Show that

$$m\frac{d\langle \vec{v}\rangle}{dt} = q\langle \vec{E}\rangle + \frac{q}{2m}\left(\langle \vec{p}\times\vec{B} - \vec{B}\times\vec{p}\rangle\right) - \frac{q^2}{m}\langle(\vec{A}\times\vec{B})\rangle.$$

$\vec{v} = \frac{1}{m}(\vec{p} - q\vec{A})$, then apply the eq. 3.71, again, but

$$\frac{\partial \vec{v}}{\partial t} = -\frac{q}{m}\frac{\partial \vec{A}}{\partial t},$$

$$\frac{d\langle \vec{v}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{\vec{v}}]\rangle - \frac{q}{m}\frac{\partial \vec{A}}{\partial t}.$$

Writing H in terms of v,

$$[\hat{H}, \hat{\vec{v}}] = [\tfrac{1}{2}mv^2 + q\varphi, \vec{v}] = \frac{m}{2}[v^2, \vec{v}] + q[\varphi, \vec{v}].$$

$$[\varphi, \vec{v}] = \frac{1}{m}[\varphi, \vec{p}]:$$

From $[f(x), p_x] = i\hbar\frac{\partial f}{\partial x}$,

$$\frac{1}{m}[\varphi, \vec{p}] = \frac{i\hbar}{m}\vec{\nabla}\varphi.$$

$$[\varphi, \vec{v}] = \frac{i\hbar}{m}\vec{\nabla}\varphi.$$

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4.59 b) con't

$[v^2, \vec{v}]$:

$$[v^2, v_x] = [v_x^2 + v_y^2 + v_z^2, v_x] = [v_y^2, v_x] + [v_z^2, v_x]$$

$$= v_y[v_y, v_x] + [v_y, v_x]v_y + v_z[v_z, v_x] + [v_z, v_x]v_z.$$

$$[v_y, v_x] = \frac{1}{m^2}[(p_y - qA_y), (p_x - qA_x)] = -\frac{q}{m^2}([A_y, p_x] + [p_y, A_x]).$$

dropping commuting terms

$$[A_y, p_x] = i\hbar \frac{\partial A_y}{\partial x}. \quad [p_y, A_x] = -i\hbar \frac{\partial A_x}{\partial y}.$$

This is the curl!

$$[v_y, v_x] = -\frac{q}{m^2} i\hbar (\nabla \times \vec{A})_z. \qquad ([v_x, v_x] = 0).$$

$$[v_z, v_x] = +\frac{q}{m^2} i\hbar (\vec{\nabla} \times \vec{A})_y.$$

$$[v^2, v_x] = \frac{i\hbar q}{m^2}\left(-v_y(\vec{\nabla} \times \vec{A})_z - (\vec{\nabla} \times \vec{A})_z v_y + v_z(\vec{\nabla} \times \vec{A})_y + (\vec{\nabla} \times \vec{A})_y v_z\right)$$

$$\vec{\nabla} \times \vec{A} = \vec{B}.$$

$$\therefore [v^2, v_x] = \frac{i\hbar q}{m^2}(-v_y B_z - B_z v_y + v_z B_y + B_y v_z).$$

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4.59 b) cont

~~$[\hat{H}, \hat{A}] =$~~ The cross product again.

$$[v^2, v_x] = \frac{i\hbar q}{m^2}\left(-(\vec{v}\times\vec{B})_x + (\vec{B}\times\vec{v})_x \right)$$

So, $$[v^2, \vec{v}] = \frac{i\hbar q}{m^2}\left(-\vec{v}\times B + \vec{B}\times\vec{v} \right).$$

$$[\hat{H}, \vec{v}] = \frac{m}{2}\left(\frac{i\hbar q}{m^2}\left(-\vec{v}\times\vec{B} + \vec{B}\times\vec{v}\right)\right) + q\left(\frac{i\hbar}{m}\vec{\nabla}\varphi \right).$$

$$\frac{d\langle\vec{v}\rangle}{dt} = \frac{i}{\hbar}\left\langle\left(\frac{i\hbar q}{2m}\left(-\vec{v}\times\vec{B} + \vec{B}\times\vec{v}\right)\right) + q\frac{i\hbar}{m}\vec{\nabla}\varphi \right)\right\rangle - \frac{q}{m}\left\langle\frac{\partial\vec{A}}{\partial t}\right\rangle.$$

$$m\frac{d\langle\vec{v}\rangle}{dt} = i^2\frac{q}{2}\left\langle\left(-\vec{v}\times\vec{B} + \vec{B}\times\vec{v}\right) + 2\vec{\nabla}\varphi\right\rangle - q\left\langle\frac{\partial\vec{A}}{\partial t}\right\rangle.$$

$$m\frac{d\langle\vec{v}\rangle}{dt} = \frac{q}{2}\left\langle\left(\vec{v}\times B - \vec{B}\times\vec{v}\right) - 2\vec{\nabla}\varphi - 2\frac{\partial A}{\partial t}\right\rangle.$$

note $-\nabla\varphi - \frac{\partial\vec{A}}{\partial t} = E.$

$$m\frac{d\langle\vec{v}\rangle}{dt} = \cancel{\frac{q}{2m}\langle(\vec{p}\times\vec{B} - \vec{B}\times\vec{p})\rangle - q}$$

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4.59 b) cont'

$$m \frac{d\langle \vec{v} \rangle}{dt} = \frac{q}{2} \langle \vec{v} \times \vec{B} - \vec{B} \times \vec{v} \rangle + q \langle \vec{E} \rangle .$$

$$\langle \vec{v} \times \vec{B} - \vec{B} \times \vec{v} \rangle = \frac{1}{m} \langle (\vec{p} - q\vec{A}) \times \vec{B} - \vec{B} \times (\vec{p} - q\vec{A}) \rangle$$

$$= \frac{1}{m} \langle \vec{p} \times \vec{B} - \vec{B} \times \vec{p} \rangle - \frac{q}{m} \langle \vec{A} \times \vec{B} - \vec{B} \times \vec{A} \rangle .$$

$$m \frac{d\langle \vec{v} \rangle}{dt} = \frac{q}{2} \left(\frac{1}{m} \langle \vec{p} \times \vec{B} - \vec{B} \times \vec{p} \rangle - \frac{q}{m} \langle \vec{A} \times \vec{B} - \vec{B} \times \vec{A} \rangle \right) + q \langle E \rangle$$

$$= \frac{q}{2m} \langle \vec{p} \times \vec{B} - \vec{B} \times \vec{p} \rangle - \frac{q^2}{2m} \langle \vec{A} \times \vec{B} - \vec{B} \times \vec{A} \rangle + q \langle E \rangle .$$

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$, because $\vec{A}$ commutes with $\vec{B}$.

$$m \frac{d\langle \vec{v} \rangle}{dt} = +q \langle E \rangle + \frac{q}{2m} \langle \vec{p} \times \vec{B} - \vec{B} \times \vec{p} \rangle - \frac{q^2}{m} \langle \vec{A} \times \vec{B} \rangle .$$

$\blacksquare$

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4.59 c) In particular, if fields $\vec{E}$ and $\vec{B}$ are uniform over the volume of the wave packet, show that

$$m \frac{d\langle \vec{v} \rangle}{dt} = q\left(\vec{E} + \langle \vec{v} \rangle \times \vec{B}\right),$$

so the expectation value of $\vec{v}$ moves according to the Lorentz force law, as we would expect from Ehrenfest's Theorem.

~~If~~ If the fields are uniform, then,

$\langle \vec{E} \rangle = \vec{E}$, ~~...~~ $\langle \vec{v} \times \vec{B} \rangle = \langle \vec{v} \rangle \times \vec{B}$,

$\langle \vec{B} \times \vec{v} \rangle = \vec{B} \times \langle \vec{v} \rangle$.

$$m \frac{d\langle \vec{v} \rangle}{dt} = \frac{q}{2}\left(\underbrace{\langle \vec{v} \rangle \times \vec{B} - \vec{B} \times \langle \vec{v} \rangle}_{\text{order } \cancel{\text{direction}} \text{ doesn't matter}}\right) + q\vec{E}$$

$$= q \langle \vec{v} \rangle \times \vec{B} + q\vec{E}.$$

■

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4.61

In classical ~~ee~~ ED the potentials $\vec{A}$ and φ are not uniquely determined; The physical quantities are the fields $\vec{E}$ and $\vec{B}$.

a) Show that the potentials

$$\varphi' = \varphi - \frac{\partial \Delta}{\partial t}, \quad \vec{A}' = \vec{A} + \vec{\nabla}\Delta,$$

(where Δ is an arbitrary real function of position and time) yield the same fields as φ and $\vec{A}$.

This equation is called a Gauge transformation, and the theory is said to be Gauge-invariant.

$$-\frac{\partial \vec{A}}{\partial t} \quad -\vec{\nabla}\varphi = \vec{E}.$$

$$-\frac{\partial A'}{\partial t} \quad -\vec{\nabla}\varphi' = -\vec{\nabla}\left(\varphi - \frac{\partial \Delta}{\partial t}\right) = -\vec{\nabla}\varphi + \frac{\vec{\nabla}\partial\Delta}{\partial t} - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\Delta$$

$$\boxed{= -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}} + \underbrace{\cancel{\vec{\nabla}\frac{\partial \Delta}{\partial t}} - \cancel{\frac{\partial}{\partial t}\vec{\nabla}\Delta}}_{\text{0 by order of partial derivatives}}.$$

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A} + \vec{\nabla}\Delta) \boxed{= \vec{\nabla}\times\vec{A}} + \underbrace{\vec{\nabla}\times\vec{\nabla}\Delta}_{\text{0 by cross product}}$$

■

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4.61

b) In quantum mechanics the potentials play a more direct role, and it is of interest to know whether the theory remains Gauge-invariant. Show that

$$\psi' \equiv e^{\frac{iq\Lambda}{\hbar}} \psi$$

satisfies the Schrödinger equation with the gauge-transformed potentials φ' and $\vec{A}'$. Since ψ' differs from ψ only by a phase factor, it represents the same physical state, and the theory is gauge-invariant.

Eq. 4.205 ~

$$i\hbar \frac{\partial \psi}{\partial t} = \underbrace{\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2 + q\varphi \right]}_{H.} \psi.$$

$$H' = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q\vec{\nabla}\Lambda\right)^2 + q\varphi - q\frac{\partial \Lambda}{\partial t}.$$

$$\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q\vec{\nabla}\Lambda\right)\psi' = \left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q\vec{\nabla}\Lambda\right) e^{\frac{iq\Lambda}{\hbar}}\psi$$

$$= \frac{\hbar}{i}\vec{\nabla} e^{\frac{iq\Lambda}{\hbar}}\psi - q\vec{A}e^{\frac{iq\Lambda}{\hbar}}\psi - q\vec{\nabla}\Lambda\, e^{\frac{iq\Lambda}{\hbar}}\psi.$$

$$= q(\nabla\Lambda)e^{\frac{iq\Lambda}{\hbar}}\psi + \frac{\hbar}{i}e^{\frac{iq\Lambda}{\hbar}}(\vec{\nabla}\psi) - q\vec{A}e^{\frac{iq\Lambda}{\hbar}}\psi - q(\vec{\nabla}\Lambda)e^{\frac{iq\Lambda}{\hbar}}\psi.$$

$$= \frac{\hbar}{i}e^{\frac{iq\Lambda}{\hbar}}\vec{\nabla}\psi - q\vec{A}e^{\frac{iq\Lambda}{\hbar}}\psi.$$

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b) con't

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$$\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q\vec{\nabla}\Lambda\right)^2 e^{\frac{iq\Lambda}{\hbar}}\psi = \left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q\vec{\nabla}\Lambda\right) \times \left(\frac{\hbar}{i}e^{\frac{iq\Lambda}{\hbar}}\vec{\nabla}\psi - q\vec{A}e^{\frac{iq\Lambda}{\hbar}}\psi\right)$$

$$= -\hbar^2\left(\frac{iq}{\hbar}(\vec{\nabla}\Lambda\cdot\vec{\nabla}\psi)e^{\frac{iq\Lambda}{\hbar}} + e^{\frac{iq\Lambda}{\hbar}}\vec{\nabla}^2\psi\right)$$

$$-\frac{\hbar q}{i}(\vec{\nabla}\cdot\vec{A})e^{\frac{iq\Lambda}{\hbar}}\psi - q^2(\vec{A}\cdot\vec{\nabla}\Lambda)e^{\frac{iq\Lambda}{\hbar}}\psi$$

$$-\frac{q\hbar}{i}e^{\frac{iq\Lambda}{\hbar}}(\vec{\nabla}\Lambda\cdot\vec{\nabla}\psi) + q^2(\vec{A}\cdot\vec{\nabla}\Lambda)e^{\frac{iq\Lambda}{\hbar}}\psi$$

$$= e^{\frac{iq\Lambda}{\hbar}}\Big[\left(-\hbar^2\nabla^2\psi + i\hbar q(\vec{\nabla}\cdot\vec{A})\psi + 2i\hbar q(\vec{A}\cdot\vec{\nabla}\psi)\right.$$

$$\left.+ q^2A^2\psi\right) - \cancel{iq\hbar\vec{\nabla}\Lambda\cdot\vec{\nabla}\psi} - \cancel{q^2(\vec{A}\cdot\vec{\nabla}\Lambda)\psi}$$

$$+ \cancel{iq\hbar\vec{\nabla}\Lambda\cdot\vec{\nabla}\psi} + \cancel{q^2(\vec{A}\cdot\vec{\nabla}\Lambda)\psi}\Big]$$

$$= e^{\frac{iq\Lambda}{\hbar}}\left(-\hbar^2\vec{\nabla}^2\psi + i\hbar q(\vec{\nabla}\cdot\vec{A})\psi + i\hbar q(\vec{A}\cdot\vec{\nabla}\psi) + q^2A^2\psi\right).$$

$$= e^{\frac{iq\Lambda}{\hbar}}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2\psi.$$

$$\therefore \left(\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}'\right)^2 + q\varphi'\right)\psi' = e^{\frac{iq\Lambda}{\hbar}}\left(\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2 + q\varphi - q\frac{\partial\Lambda}{\partial t}\right)\psi$$

$$= e^{\frac{iq\Lambda}{\hbar}}\left(i\hbar\frac{\partial\psi}{\partial t} - q\frac{\partial\Lambda}{\partial t}\psi\right) = i\hbar\frac{\partial}{\partial t}\left(e^{\frac{iq\Lambda}{\hbar}}\psi\right) = i\hbar\frac{\partial\psi'}{\partial t}.$$

$\blacksquare$.

~~Other~~
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Neutron $\qquad \mu_n = -1.91\,\mu_N$.

μ_N the nuclear magneton.

The operator to measure $\hat{\mu} = \mu_n \vec{\sigma} = \gamma_n \hat{S}$.

The gyromatic ratio $\gamma_n = \dfrac{g_n \mu_N}{\hbar}$, with dimensionless $g_n = \dfrac{2\mu_n}{\mu_N} = -3.826$

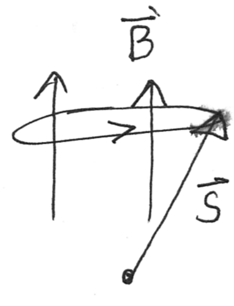
Along z, $\mu_z = g_s \mu_N m$.

$$(\hat{x}' \;\; \hat{y}') = (\hat{x} \;\; \hat{y}) \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix},$$

the transformation to a convenient, rotating frame of reference.

$$\frac{d\vec{S}}{dt} = \vec{\tau} = \vec{\mu}_n \times \vec{B} = \gamma_n \vec{S} \times \vec{B},$$

$\hat{z} B_0$ is a constant magnetic field that induces larmour precession.



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a) Solve classical equations of motion,

$$\dot{\vec{S}} = \frac{d\vec{S}}{dt} = \gamma_n \vec{S} \times \vec{B}.$$

$\vec{B} = B_0 \hat{z}$, for Larmour precession.

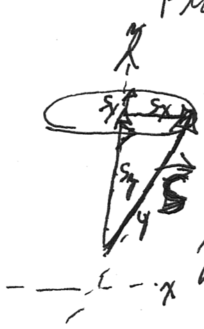
$\dot{S}_z = 0.$

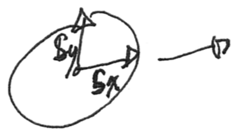
From the cross product,

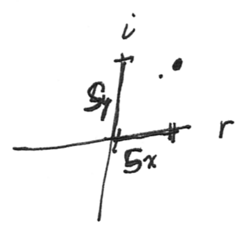
$\dot{S}_x = \gamma_n B S_y$, and

$\dot{S}_y = -\gamma_n B S_x.$

Projected on $\mathbb{C}$,







$z = S_x + i S_y = S_+ .$

$$\dot{S}_+ = \gamma_n B S_y - i\gamma B S_x = \gamma_n B(S_y - i S_x) = \gamma_n B i\left(\frac{S_y}{i} - S_x\right)$$

$$= \gamma_n B i(-S_x - i S_y) = -i\gamma_n B(S_x + i S_y).$$

$$\dot{S}_+ = -i\gamma_n B S_+$$

The soln to this diffeq is

$\boxed{S_+ = S_+^\circ e^{-i\gamma_n B t}}$, where $\gamma_n B$ is the frequency of ~~the~~ Larmour precession.

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a) con't

Solve for the same using the operator $\vec{\omega}\, dt \times$!

$$\frac{d\vec{S}}{dt} = \frac{d\vec{S}'}{dt} + \vec{\omega} \times \vec{S}.$$

$\frac{d\vec{S}'}{dt}$ is the precession of $\vec{S}$ relative to the rotating references frame, which of course will be 0.

The original equation is now

$$\frac{d\vec{S}'}{dt} + \vec{\omega} \times \vec{S} = \gamma_n \vec{S} \times \vec{B}.$$

$\vec{\omega} = \omega \hat{z}$, where $\vec{\omega}$ is the angular velocity of the rotating reference frame.

So clearly $\dot{S}_z = 0$ hasn't changed.

From the cross product,

$$\dot{S}_x - \omega S_y = \gamma_n B S_y, \text{ and}$$

$$\dot{S}_y + \omega S_x = -\gamma_n B S_x.$$

Equations of motion are,

$\dot{S}_x = (\gamma_n B + \omega) S_y$, and

$\dot{S}_y = -(\gamma_n B + \omega) S_x$.

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a) con't

Clearly, the new solution is

$$S_+ = S_+^\circ e^{-i(\gamma_n B + \omega)t} .$$

The exponential rotation has the frequency $\gamma_n B + \omega$, which is zero when

$\omega = -\gamma_n B$.

Another way of capturing this is

$$S_+ = S_+^\circ e^{-i\gamma_n \left(B + \frac{\omega}{\gamma_n} \right) t},$$

where we call $B + \frac{\omega}{\gamma_n}$ the effective field B'.

So, we see that when the frame of reference rotates at $\omega = -\gamma_n B$, the spin does not precess relative to the frame, and S° is a constant. The spin still precesses, of course, and if we transformed back to the non-rotating frame, we would find the result of that precession.

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b)

~~Other~~
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Solve this system quantum mechanically, using the spin state $\chi_0 = \begin{pmatrix} a \\ b \end{pmatrix}$

Show that the expected value of the spin $\langle S \rangle (t)$ agrees with the classical result.

From the class notes we see that the Larmor precession is associated with the phase of the state, ~~...~~ since there are only fixed measurement values of $\hat{S}$.

$$H = -\gamma_n B S_z \quad (\text{from } -\vec{\mu} \cdot \vec{B})$$

$$= \underbrace{-\frac{\gamma_n B \hbar}{2}}_{\text{Energy } E} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The state $\chi_0 = \begin{pmatrix} a \\ b \end{pmatrix}$ is a superposition of states $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The eigenvalues are $E_\pm = \mp \frac{\gamma_n B \hbar}{2}$.

So, the time-dependant solution to the Schrodinger Eq is

$$\chi(t) = a \chi_+ e^{-\frac{iE_+ t}{\hbar}} + b \chi_- e^{-\frac{iE_- t}{\hbar}}$$

$$= \begin{pmatrix} a e^{-\frac{iE_+ t}{\hbar}} \\ b e^{-\frac{iE_- t}{\hbar}} \end{pmatrix} = \begin{pmatrix} a e^{i\frac{\gamma_n B}{2}} \\ b e^{-i\frac{\gamma_n B}{2}} \end{pmatrix}.$$

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b) cont

So we have $\chi(t)$ with its initial state $\chi(0)$.

Normalizing a and b,

$$|a|^2 + |b|^2 = 1,$$

gives a solution $a = \cos(\frac{\alpha}{2})$ and $b = \sin(\frac{\alpha}{2})$.

$$\chi(t) = \begin{pmatrix} \cos(\frac{\alpha}{2}) e^{i \frac{\gamma_n B}{2} t} \\ \sin(\frac{\alpha}{2}) e^{-i \frac{\gamma_n B}{2} t} \end{pmatrix}, \quad \text{with } \chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}.$$

The expectation values may now be computed.

$$\langle S_x \rangle = \langle \chi | S_x | \chi \rangle = \frac{\hbar}{2} \chi^{\dagger}(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \chi(t).$$

$$\langle S_x \rangle = \frac{\hbar}{2} \left(\sin\frac{\alpha}{2} e^{+i \frac{\gamma_n B}{2} t} \quad \cos\frac{\alpha}{2} e^{-i \frac{\gamma_n B}{2} t} \right) \chi(t).$$

$$\langle S_x \rangle = \frac{\hbar}{2} \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} e^{+i \frac{\gamma_n B}{2} t} + \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} e^{-i \gamma_n B t}.$$

$$\langle S_x \rangle = \frac{\hbar}{2} 2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} \left(e^{i \gamma_n B t} + e^{i \gamma_n B t} \right).$$

Using a trig identity and the complex definition of cos,

$$\langle S_x \rangle = \frac{\hbar}{2} \sin(\alpha) \cos(\gamma B t).$$

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b) con't

$\langle S_y \rangle$ comes out similarly but with $S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

$$\langle S_y \rangle = -\frac{\hbar}{2} \sin\alpha \sin(\gamma_n B t).$$

$\langle S_z \rangle$ we expect to be non-changing.

$$\langle S_z \rangle = \chi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \chi$$

$$= \frac{\hbar}{2}\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) = \frac{\hbar}{2}\cos\alpha$$ by trig identity.

$\langle S_z \rangle$ therefore does not depend on t and is a constant of motion.

The chapter 4 result #56 is

$$f(\phi + \varphi) = e^{iL_z \varphi/\hbar} f(\phi).$$

The generator of rotation about the z-axis is $\frac{L_z}{\hbar}$. ϕ is an arbitrary angle which I associate with the rotation of the coordinate system ∴

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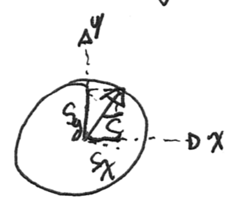
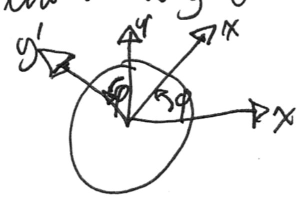
b) con't

In the xy plane:

In the rotating frame



the rotation of L is generated according to $f(\phi+\varphi) = e^{i\frac{L_z}{\hbar}\varphi} f(\phi)$,

where $f(\phi)$ is the original function in the non-rotating coordinate system.

For the spin state, $f(\phi)$ = one of the three expectation values,

$\langle S_z \rangle = \frac{\hbar}{2}\cos\alpha$ (a constant of motion $\alpha \equiv$ const.),

$\langle S_y \rangle = -\frac{\hbar}{2}\sin\alpha \sin(\gamma_n B t)$, and

$\langle S_x \rangle = +\frac{\hbar}{2}\sin\alpha\cos(\gamma_n B t)$.

I associate the angle ϕ with $\gamma_n B t$:

$\phi = \gamma_n B t$.

The expectation values in the rotating frame of reference are then,

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b) cont

~~Otto~~

HW14

$$\langle S_z \rangle' = \frac{\hbar}{2} \cos\alpha ,$$

$$\langle S_y \rangle' = -\frac{\hbar}{2} \sin\alpha \sin(\gamma_n B t)\, e^{i \frac{S_z}{\hbar} \varphi}, \text{ and}$$

$$\langle S_x \rangle' = \frac{\hbar}{2} \sin\alpha \cos(\gamma_n B t)\, e^{i \frac{S_z}{\hbar} \varphi}.$$

We've assumed spin accounts for all angular momentum so that $L_z = S_z$.

The state $\chi(t)'$ represents the state $\chi(t)$ in the rotating frame of reference.

If the frame of reference rotates at the angular velocity ω, it clears an angle φ in ~~ω~~t seconds:

$$\varphi = \omega t .$$

By Griffith's result,

$$\chi(t)' = e^{i \frac{L_z}{\hbar} \omega t} \chi(t).$$

again, $L_z = S_z$, and in this case it is equivalent to (with the additional ω factor) because $\vec{\omega}$ is $\vec{\omega} \cdot \vec{S}$ completely along $\hat{z}$.

$$\chi(t)' = e^{i \frac{\vec{\omega} \cdot \vec{S} t}{\hbar}} \chi(t) = e^{i \frac{S_z}{\hbar} \omega t} \chi(t).$$

$$\boxed{\therefore\ \chi(t) = e^{-i \frac{S_z}{\hbar} \omega t} \chi(t).}$$

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b) cont

The original Hamiltonian is (in the fixed frame),

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma_n B S_z = -\gamma_n B \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we had an effective field

$$\vec{B}' = \vec{B} + \frac{\vec{\omega}}{\gamma_n}$$

the Hamiltonian

$$H = -\vec{\mu}\cdot\vec{B}' = -\vec{\mu}\cdot\vec{B} + \vec{\mu}\cdot\frac{\vec{\omega}}{\gamma_n}.$$

assuming $\vec{\omega} = \omega\hat{z}$, with $\vec{B} = B\hat{z}$,

$$H = -\gamma_n B S_z + \frac{\gamma_n \omega S_z}{\gamma_n} = -\gamma_n B S_z + \omega S_z.$$

$H = -\gamma_n\left(B + \frac{\omega}{\gamma_n}\right) S_z$, which is similar to the hamiltonian for a spin particle in a Stern-Gerlach machine.

The eigenvalues are now $E_\pm = \mp\gamma_n\left(B + \frac{\omega}{\gamma_n}\right)$.

The states are still,

$$\chi(t)' = a e^{\frac{-iE_+ t}{\hbar}} + b e^{\frac{-iE_- t}{\hbar}},$$ where

we would normalize a and b the same way,

but the exponential expands to

$$\chi(t)' = a e^{+i\frac{\gamma_n B}{\hbar}t} e^{+i\omega\frac{t}{\hbar}} + b e^{-i\frac{\gamma_n B}{\hbar}t} e^{-i\omega\frac{t}{\hbar}},$$

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b) con't

So, we can see this is equivalent to operating on the state with

$$\boxed{\chi(t) = e^{-i\omega S_z \frac{t}{\hbar}} \chi'(t),}$$

where the $\mp$ remains consistent when acting on the χ_- and χ_+ states.

c)

$B_0 \hat{z}$ is a holding field for the spin state m.

$B_1(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$ will cause the state to transition.