3.7) Operator $\hat{Q}$ has eigenfunctions $f(x)$ and $g(x)$, with eigenvalues q.

a) Show any linear combo of f and g are eigenfunctions of $\hat{Q}$ with eigenvalue q.

$$\hat{Q}|f\rangle = q|f\rangle. \qquad \hat{Q}|g\rangle = q|g\rangle.$$

$$\hat{Q}\left(c_1|f\rangle + c_2|g\rangle\right) = \hat{Q}\cdot c_1|f\rangle + \hat{Q}c_2|g\rangle = c_1(\hat{Q}|f\rangle) + c_2(\hat{Q}|g\rangle)$$

$$= c_1 q|f\rangle + c_2 q|g\rangle = q(c_1|f\rangle + c_2|g\rangle).$$

$$\hat{Q}(c_1|f\rangle + c_2|g\rangle) = q(c_1|f\rangle + c_2|g\rangle).$$

Q.E.D.

b) Are $f(x)=\exp(x)$ and $g(x)=\exp(-x)$ eigenfunctions of the operator $\frac{d^2}{dx^2}$, with identical eigenvalues

$$\hat{Q} = \frac{d^2}{dx^2} \qquad \hat{Q}|f\rangle = \frac{d^2}{dx^2}e^{x} = \underset{\text{eigenvalue}}{(1)}e^{x} \qquad \hat{Q}|g\rangle = \frac{d^2}{dx^2}e^{-x} = \underset{\text{eigenvalue}}{(-1)^2}e^{-x}$$

Satisfies Sturm-Liouville, so are eigenfunctions.

Same eigenvalues.

$\overbrace{c_1\exp(x) + c_2\exp(-x)}^{\alpha}$ and $\overbrace{D_1\exp(x) + D_2\exp(-x)}^{\beta}$ are distinct linear combinations of the eigenfunctions. To be orthogonal,

$$\int_{-1}^{1} \alpha\beta\, dx = 0 = \int_{-1}^{1} (c_1\exp(x) + c_2\exp(-x))(D_1\exp(x) + D_2\exp(-x))\, dx$$

This is going to be complicated, but there is an obvious case which should be sufficient, namely, sinh and cosh, which are <u>orthogonal</u> functions, and in terms of $f(x)$ and $g(x)$

$\sinh(x) = \frac{1}{2}(f(x) - g(x))$ and $\cosh(x) = \frac{1}{2}(f(x) + g(x))$, with $\sinh \perp \cosh$.

$\boxed{3.10}$ Is the ground state of the infinite square well an eigenfunction of momentum? If so, what is its momentum? If not, why not?

$|1\rangle \rightarrow$ ground state of I.S.W.

Position space representation is $\psi(x) = A\cos\left(\frac{\pi x}{a}\right)$ before normalization determines A.

$\hat{p} = -i\hbar \frac{d}{dx} \Rightarrow \hat{p}\,\psi \stackrel{?}{=} p\,\psi$, if satisfied, then momentum has the eigenfunction in question.

$$-i\hbar \frac{d}{dx}\left(A\cos\left(\frac{\pi x}{a}\right)\right) = +i\hbar A \frac{\pi}{a} \sin\left(\frac{\pi x}{a}\right).$$

$\boxed{\text{No}}$ After acting on the state, the operator <u>does not</u> return the same eigenstate with only an eigenvalue multiplied on front.

3.13 Prove the following commutator identity:

$$[AB, C] = A[B,C] + [A,C]B$$

a)

$$[AB, C] = ABC - CAB, \text{ from definition.}$$

$$ABC - CAB = ABC - CAB + ACB - ACB.$$

$$ABC - ACB = A[B,C].$$

$$ACB - CAB = [A,C]B.$$

$$\therefore [AB,C] = A[B,C] + [A,C]B.$$

$\blacksquare$

b) Show $[x^n, p] = i\hbar n x^{n-1}$!

Need to use a test function f,

$$[x^n, p]f = (x^n)\left(-i\hbar \frac{d}{dx}\right)f + \left(i\hbar \frac{d}{dx}\right)(x^n) f$$

$$= i\hbar\left(-x^n \frac{df}{dx} + n x^{n-1} f + x^n \frac{df}{dx}\right)$$

$$= i\hbar n x^{n-1} f.$$

$$\therefore [x^n, p] = i\hbar n x^{n-1}. \quad \blacksquare$$

c) Show $[f(x), p] = i\hbar \frac{df}{dx}$

$$[f(x), p]\, g = \cancel{-f i\hbar \frac{dg}{dx}} + i\hbar \frac{df}{dx} g + \cancel{i\hbar f \frac{dg}{dx}}$$

$$= i\hbar \frac{df}{dx} g.$$

$$\therefore [f, p] = i\hbar \frac{df}{dx}.$$

$\blacksquare$

3.14) Prove $\sigma_x \sigma_H \geq \frac{\hbar}{2m}|\langle p \rangle|$.

Right off, a stationary state has a constant energy in time, i.e., is an eigenstate of the Hamiltonian, and therefore commutes with the Hamiltonian.

$$H = \frac{p^2}{2m} + V(x).$$

0, since a function of x commutes with x.

$$[x, H] = [x, \tfrac{p^2}{2m} + V(x)] = [x, \tfrac{p^2}{2m}] + \cancel{[x, V(x)]}.$$

Recall $[x, p] = i\hbar$.

$$[x, \tfrac{p^2}{2m}] = \frac{1}{2m}[x, p^2] = ([x,p]p + p[x,p])\frac{1}{2m} = \frac{1}{2m}(i\hbar p + i\hbar p) = \frac{i\hbar p}{m}.$$

Using the Generalized Uncertainty Principle,

$$\sigma_x^2 \sigma_H^2 \geq \left(\frac{1}{2i}\langle [x, H] \rangle\right)^2,$$

$$\sigma_x^2 \sigma_H^2 \geq \left(\frac{1}{2i}\left\langle \frac{i\hbar p}{m} \right\rangle\right)^2 = \left(\frac{\hbar}{2m}\langle p \rangle\right)^2.$$

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m}|\langle p \rangle|.$$

■

For stationary states, the probability distribution in position is constant. Therefore, $\langle p \rangle = 0$, and so this uncertainty relationship may take any value greater than zero.

Also, $\sigma_H = 0$, so this really just says $0 \geq 0$.

3.15 Show that two non-commuting operators cannot have a complete set of common eigenstates.
Hint: Show that if $\hat{P}$ and $\hat{Q}$ have a complete set of common eigenfunctions, then $[\hat{P}, \hat{Q}]f = 0$ for any function in Hilbert space.

This is a proof from contradiction, then.

assuming $\hat{P}$ and $\hat{Q}$ share a complete set of eigenfunctions f_n,

$$\hat{P} f_n = p f_n \text{ and } \hat{Q} f_n = q f_n,$$

and any function in Hilbert space may be represented as a linear combination $f = \sum c_n f_n$.

$$[\hat{P}, \hat{Q}] f = (\hat{P}\hat{Q} - \hat{Q}\hat{P}) \sum c_n f_n = \hat{P}\hat{Q} \sum c_n f_n - \hat{Q}\hat{P} \sum c_n f_n$$

$$= \hat{P} \sum c_n q_n f_n - \hat{Q} \sum c_n p_n f_n = \sum c_n p_n q_n f_n - \sum c_n q_n p_n f_n = 0.$$

I've not restricted the functional form of f, so for any function f,

$$[\hat{P}, \hat{Q}] f = 0.$$

$\therefore$ the operators must commute if they have a common complete set of eigenfunctions.

$\boxtimes$

4.18 The raising and lowering operators change the value of m by one unit:

$$L_\pm f_\ell^m = (A_\ell^m) f_\ell^{m\pm1},$$

where A_ℓ^m is some constant.

Question: What is A_ℓ^m, if the eigenfunctions are to be normalized?

Hint: First show that $L_\mp$ is the hermitian conjugate of $L_\pm$ (since L_x and L_y are observables, you may assume they are hermitian, but prove it if you like), then use Eq. 4.112.

Answer: $A_\ell^m = \hbar\sqrt{\ell(\ell+1)-m(m\pm1)} = \hbar\sqrt{(\ell\mp m)(\ell\pm m+1)}$

Note what happens at the top and bottom of the ladder (i.e. when you apply L_+ to f_ℓ^ℓ or L_- to $f_\ell^{-\ell}$.

To prove $L_\mp = L_\pm^\dagger$, I apply the operator to a state g, and attempt to modify it to apply to the row vector f, where it would appear as its complex conjugate.

$$\langle f|g\rangle \to \langle f|(L_\pm|g\rangle) = \langle f|(L_x|g\rangle \pm iL_y|g\rangle).$$

Note: $L_x = L_x^\dagger$
$L_y = L_y^\dagger$
but $\pm \to \mp$ for complex conjugate.

$$= (\langle f|L_x)|g\rangle \mp i(\langle f|L_y)|g\rangle$$

$$= (\langle f|(L_x \mp iL_y))|g\rangle$$

$$= (\langle f|L_\mp)|g\rangle.$$

$$\therefore\ L_\mp = L_\pm^\dagger.$$

Eq. 4.112: $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$. Use this to find the normalized value of A_ℓ^m.

4.112 implies $L_\mp L_\pm = L^2 - L_z^2 \pm \hbar L_z$.

Using $L^2|f_\ell^m\rangle = (\hbar^2\ell(\ell+1))|f_\ell^m\rangle$
$L_z|f_\ell^m\rangle = \hbar m|f_\ell^m\rangle$

$$\langle f_\ell^m|L_\mp L_\pm|f_\ell^m\rangle = \langle f_\ell^m|(L^2 - L_z^2 \pm \hbar L_z)|f_\ell^m\rangle = \langle f_\ell^m|(\hbar^2\ell(\ell+1) - \hbar^2 m^2 \pm \hbar^2 m)|f_\ell^m\rangle$$

4.18 cont

$$= \hbar^2 (l(l+1) - m^2 \pm m) \langle f_l^m | f_l^m \rangle$$

$$= \hbar^2 (l(l+1) - m(m \pm 1)).$$

$$\therefore (A_l^m)^2 = \hbar^2 (l(l+1) - m(m \pm 1)), \text{ and}$$

$$A_l^m = \hbar \sqrt{l(l+1) - m(m \pm 1)}.$$

This matches the given answer.

$\blacksquare$

4.19

a) Starting with the canonical commutation relations for position and momentum, work out the following commutators:

$$[L_z, x] = i\hbar y,\ [L_z, y] = -i\hbar x,\ [L_z, z] = 0,$$

$$[L_z, p_x] = i\hbar p_y,\ [L_z, p_y] = -i\hbar p_x,\ [L_z, p_z] = 0.$$

$$[L_z, x] = [xp_y - yp_x, x] = [xp_y, x] - [yp_x, x] = 0 - y[p_x, x]$$

$$\boxed{= i\hbar y.}$$

$$[L_z, y] = [xp_y - yp_x, y] = [xp_y, y] - [yp_x, y] = x[p_y, y] - 0$$

$$\boxed{= -i\hbar x}$$

$[L_z, z] = [xp_y - yp_x, z] = \boxed{0}$ (nothing here that could possibly not commute.)

$$[L_z, p_x] = [xp_y - yp_x, p_x] = [xp_y, p_x] - [yp_x, p_x] = p_y[x, p_x] - 0$$

$$= \boxed{i\hbar p_y}$$

$$[L_z, p_y] = [xp_y - yp_x, p_y] = [xp_y, p_y] - [yp_x, p_y] = 0 - i\hbar p_x$$

$$\boxed{= -i\hbar p_x}$$

$[L_z, p_z] = [xp_y - yp_x, p_z] = 0$, again, nothing here that might possibly not commute.

4.19 con't

b) Use these results to obtain $[L_z, L_x] = i\hbar L_y$ directly from Eq. 4.96.

4.96 just gives the "L's in terms of "p"s and "x,y,z".

$$[L_z, L_x] = [L_z, yp_z - zp_y] = [L_z, yp_z] - [L_z, zp_y] = [L_z, y]p_z - [L_z, p_y]z$$

$$= -i\hbar x p_z + i\hbar p_x z = i\hbar(-xp_z + zp_x) = i\hbar L_y.$$

c) Evaluate the commutators $[L_z, r^2]$ and $[L_z, p^2]$ (where $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$).

$$[L_z, r^2] = [L_z, x^2] + [L_z, y^2] + [L_z, z^2] = [L_z, x]x + x[L_z, x] + [L_z, y]y + y[L_z, y] + 0$$

$$= i\hbar y x + x i\hbar y + (-i\hbar x)y + y(-i\hbar x) = \boxed{0}.$$

$$[L_z, p^2] = [L_z, p_x]p_x + p_x[L_z, p_x] + [L_z, p_y]p_y + p_y[L_z, p_y] + 0$$

$$= (i\hbar p_y)p_x + p_x(i\hbar p_y) + (-i\hbar p_x)p_y + p_y(-i\hbar p_x)$$

$$= \boxed{0}.$$

d) Show that the hamiltonian $H = \frac{p^2}{2m} + V$ commutes with all three components of $\vec{L}$, provided V only depends on r. (Thus H, L^2, and L_z are mutually compatible observables.)

The hamiltonian contains only the operators p^2 and r. I have already proved that L_z commutes with these operators. By cyclic permutation, the same arguments apply to L_x and L_y. Q.E.D.