Two-Dimensional Quantum Harmonic Oscillator

2D Quantum Harmonic Oscillator

- in ch5, Schrödinger constructed the coherent state of the 1D H.O. to describe a classical particle with a wave packet whose center in the time evolution follows the corresponding classical motion
- the H.O. plays a significant role in demonstrating the concept of quantum-classical correspondence ∵ it can be analytically solved in both CM & QM
- the Schrödinger coherent state of the 2D H.O. is a nonspreading wave packet with its center moving along the classical trajectories
- we will start from the time-dep. Schrödinger coherent state for 2D H.O. to extract the stationary coherent states that are localized on the corresponding classical trajectories

Eigenstates of the 2D Isotropic Harmonic Oscillator

- the Hamiltonian for the isotropic 2D H.O. in Cartesian coordinate：

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2)$$

the time-indep Schrödinger eq. is：

$$\left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} m\omega^2 (x^2 + y^2) \right] \psi(x, y) = E\,\psi(x, y)$$

- $\psi(x, y)$ is separable： $\psi(x, y) = \mathrm{X}(x)\mathrm{Y}(y)$

$$\rightarrow \quad \frac{1}{\mathrm{X}(x)} \left(-\frac{\hbar^2}{2m} \frac{d^2}{d\,x^2} + \frac{1}{2} m\omega^2 x^2 \right) + \frac{1}{\mathrm{Y}(y)} \left(-\frac{\hbar^2}{2m} \frac{d^2}{d\,y^2} + \frac{1}{2} m\omega^2 y^2 \right) = E$$

Eigenstates of the 2D Isotropic Harmonic Oscillator

- consequently, we have obtained 2 differential eq. for the 1D H.O. :

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{d\,x^2}+\frac{1}{2}m\omega^2x^2\right)\mathrm{X}(x)=E^x\ \mathrm{X}(x)$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{d\,y^2}+\frac{1}{2}m\omega^2y^2\right)\mathrm{Y}(y)=E^y\ \mathrm{Y}(y)$$

where $E^x+E^y=E$

- the eigenfunction and the eigenvalue of the 2D isotropic H.O. are given

by $\tilde{\psi}_{m,n}(\xi_x,\xi_y)=\left(2^{n+m}m\ !n\ !\cdot\pi\right)^{-1/2}\ e^{-(\xi_x^2+\xi_y^2)/2}H_m(\xi_x)H_n(\xi_y)$

$$E_{m,n}=(m+n+1)\,\hbar\omega$$

where $\xi_x=\sqrt{m\omega/\hbar}\,x$ & $\xi_y=\sqrt{m\omega/\hbar}\,y$

Eigenstates of the 2D Isotropic Harmonic Oscillator

- the eigenvalues of the 2D H.O. are the *sum* of the two 1D oscillator eigenenergies & the eigenfunctions are the *product* of two 1D eigenfunctions

- It can be found that the eigenstates in

$$\tilde{\psi}_{m,n}(\xi_x,\xi_y)=\left(2^{n+m}m!n!\cdot\pi\right)^{-1/2}\ e^{-(\xi_x^2+\xi_y^2)/2}H_m(\xi_x)H_n(\xi_y)$$

do *not* reveal the characteristics of classical elliptical trajectories even in the correspondence limit of large quantum number

Eigenstates of the 2D Isotropic Harmonic Oscillator

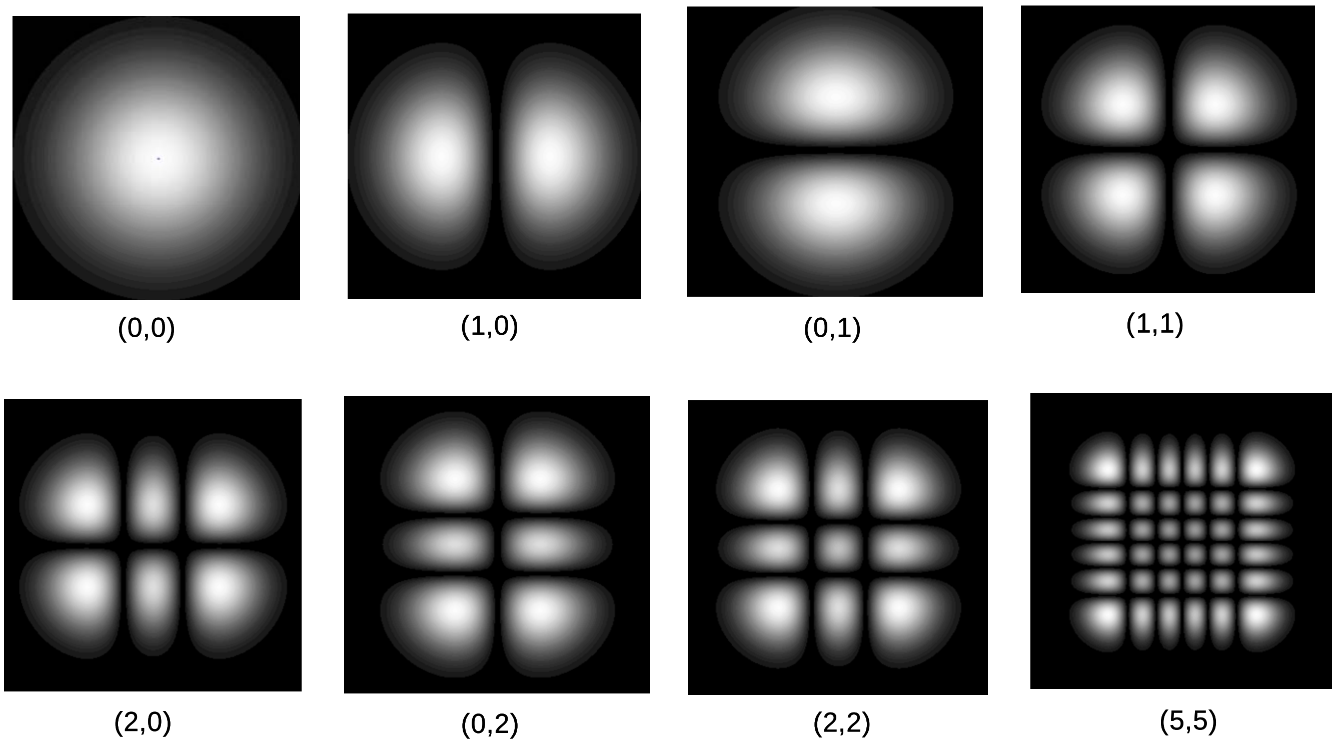


Figure 7.1 Probability density patterns of eigenstates for the 2D isotropic harmonic oscillator

Stationary Coherent States of the 2D Isotropic H.O.

- It is clear that the center of the wave packet follows the motion of a classical 2D isotropic harmonic oscillator, i.e.,

$$\xi_x = \sqrt{2}\alpha_x \cos(\omega t - \phi_x) \; ; \; \xi_y = \sqrt{2}\alpha_y \cos(\omega t - \phi_y)$$

- The Schrödinger coherent state for the 2D isotropic harmonic oscillator is a product of two infinite series. The method of the triangular partial sums is used to make precise sense out of the product of two infinite series.

- Mathematically, the notion of triangular partial sums is called the Cauchy product of the double infinite series

Stationary Coherent States of the 2D Isotropic H.O.

- With the representation of the Cauchy product, the terms can be arranged diagonally by grouping together those terms for which has a fixed value:

$$\Psi(\xi_x,\xi_y,t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_x\ e^{i\phi_x})^m(\alpha_y\ e^{i\phi_y})^n}{\sqrt{m\ !}\ \sqrt{n\ !}}e^{-(\alpha_x^2+\alpha_y^2)/2}\ \ \psi_{m,n}(\xi_x,\xi_y)\ \ e^{-i(m+n+1)\omega\, t}$$

$$=\sum_{N=0}^{\infty}\sum_{K=0}^{N}\frac{(\alpha_x\ e^{i\phi_x})^K(\alpha_y\ e^{i\phi_y})^{N-K}}{\sqrt{K\ !}\ \sqrt{(N-K)\ !}}e^{-(\alpha_x^2+\alpha_y^2)/2}\ \ \psi_{K,N-K}(\xi_x,\xi_y)\ \ e^{-i(N+1)\omega\, t}$$

$$=\left(\sum_{N=0}^{\infty}e^{-(\alpha_x^2+\alpha_y^2)/2}e^{-i(N+1)\omega\, t}\frac{(\alpha_y\ e^{i\phi_y})^N}{\sqrt{N\ !}}\right.$$

$$\left.\times\sum_{K=0}^{N}\frac{\sqrt{N\ !}}{\sqrt{K\ !\ (N-K)\ !}}\left[\frac{\alpha_x}{\alpha_y}e^{i(\phi_x-\phi_y)}\right]^K\ \ \psi_{K,N-K}(\xi_x,\xi_y)\right)$$

Stationary Coherent States of the 2D Isotropic H.O.

- After some algebra,

$$\Psi(\xi_x,\xi_y,t)=\sum_{N=0}^{\infty} C_N\,\Phi_N(\xi_x,\xi_y;A,\phi)\,e^{-i(N+1)\omega t}$$

$$C_N=e^{-(\alpha_x^2+\alpha_y^2)/2}\,\frac{\left(\sqrt{1+A^2}\,\alpha_y\,e^{i\phi_y}\right)^N}{\sqrt{N!}}$$

$$\Phi_N(\xi_x,\xi_y;A,\phi)=\frac{1}{\left(\sqrt{1+A^2}\right)^N}\sum_{K=0}^{N}\binom{N}{K}^{1/2}(Ae^{i\phi})^K\,\tilde{\psi}_{K,\,N-K}(\xi_x,\xi_y)$$

$$A=\frac{\alpha_x}{\alpha_y},\qquad \phi=\phi_x-\phi_y$$

- The wave function above represents a type of normalized stationary coherent state.

Stationary Coherent States of the 2D Isotropic H.O.

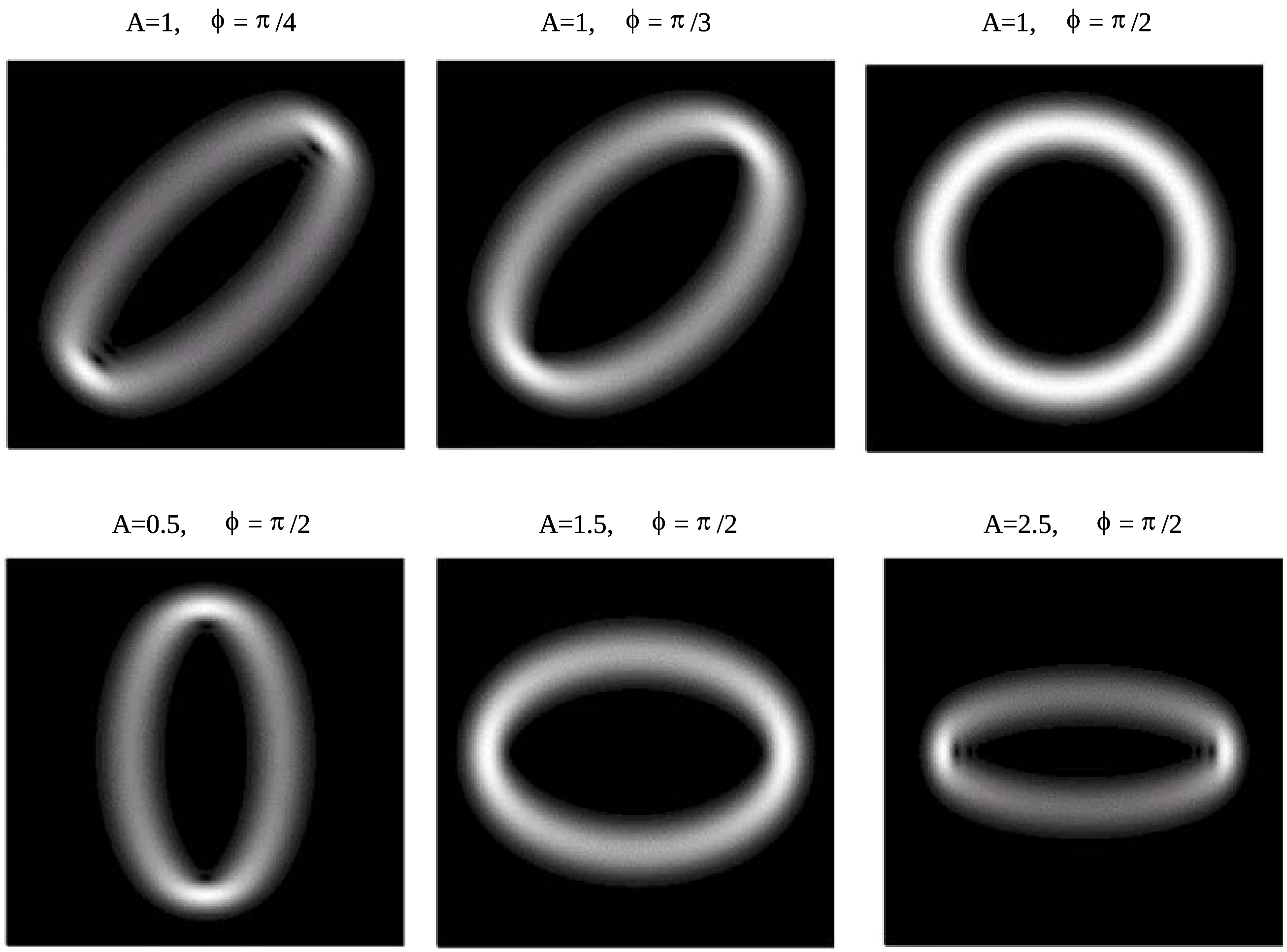


Figure 7.2 Wave patterns of the stationary coherent states $\left|\Phi_N(\xi_x, \xi_y; A, \phi)\right|^2$

for *N*=32 with different values of the parameters *A* and ϕ .

Stationary Coherent States of the 2D Isotropic H.O.

- It can be seen that the coherent states $\Phi_N(\xi_x, \xi_y; A, \phi)$ correspond to the elliptic stationary states.
- The superposition of two elliptic states with a phase factor ϕ in the opposite sign can form a standing wave pattern:

$$\Phi_N(\xi_x, \xi_y; A, \phi) \pm \Phi_N(\xi_x, \xi_y; A, -\phi)$$

- Next figure shows the standing wave patterns corresponding to the elliptic states shown in figure above.

Stationary Coherent States of the 2D Isotropic H.O.

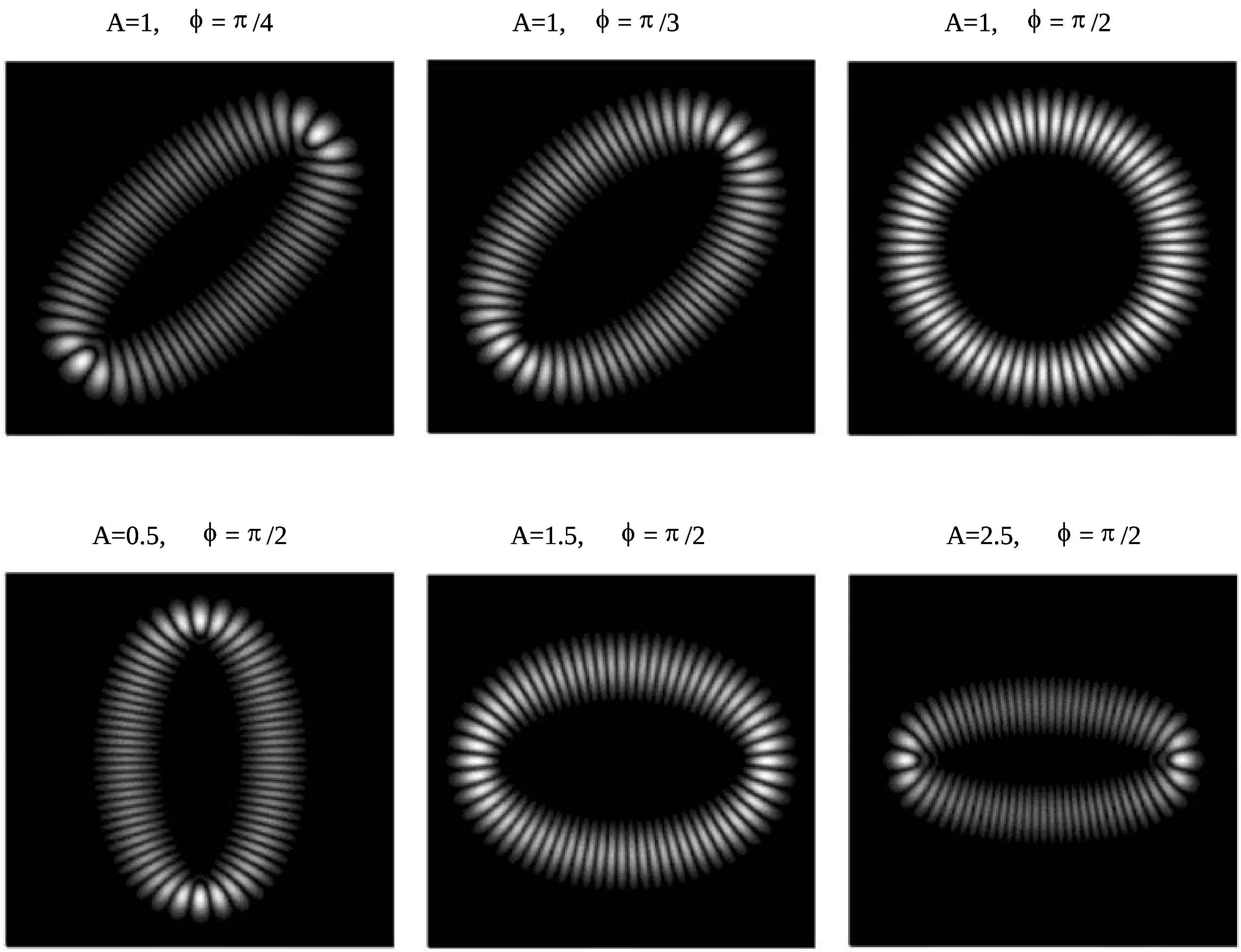


Figure 7.3 Standing wave patterns corresponding to the elliptic states shown in figure 7.2.

Stationary Coherent States of the 2D Isotropic H.O.

- $\Psi(\xi_x,\xi_y,t)=\sum_{N=0}^{\infty} C_N\,\Phi_N(\xi_x,\xi_y;A,\phi)\,e^{-i(N+1)\omega t}$ manifestly reveals the relationship between the Schrödinger coherent state and the stationary coherent state.
- $\left|C_N\right|^2$ represents the probability of finding the system in the elliptic stationary state with order *N*.

$$\left|C_N\right|^2=\frac{\left(\alpha_x^2+\alpha_y^2\right)^N}{\sqrt{N!}}e^{-(\alpha_x^2+\alpha_y^2)}$$

- The probability distribution is identical to the Poisson distribution with the mean value of $<N>=\alpha_x^2+\alpha_y^2$

Angular Momentum in 2D Confined Systems

- angular momentum of a classical particle is a vector quantity, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- Angular momentum is the property of a system that describes the tendency of an object spinning about the point *r* = 0 to remain spinning, classically.
- For the motion of a classical 2D isotropic harmonic oscillator, the angular momentum about the *z*-axis can be found to be independent of time:

$$\begin{cases} x(t) = \sqrt{\dfrac{\hbar}{m\omega}}\sqrt{2}\,|\alpha_x|\cos(\omega t - \phi_x) \\ y(t) = \sqrt{\dfrac{\hbar}{m\omega}}\sqrt{2}\,|\alpha_y|\cos(\omega t - \phi_y) \end{cases} \qquad \begin{cases} p_x(t) = m\dfrac{d\,x(t)}{d\,t} = -\sqrt{m\omega\hbar}\sqrt{2}\,|\alpha_x|\sin(\omega t - \phi_x) \\ p_y(t) = m\dfrac{d\,y(t)}{d\,t} = -\sqrt{m\omega\hbar}\sqrt{2}\,|\alpha_y|\sin(\omega t - \phi_y) \end{cases}$$

$$x(t)\,p_y(t) - y(t)\,p_x(t) = -\sqrt{2}\,\hbar\,|\alpha_x||\alpha_y|\sin(\phi_x - \phi_y)$$

Angular Momentum in 2D Confined Systems

- In quantum mechanics, the angular momentum is associated with the operator $\hat{\mathbf{L}}$, that is defined as $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$
- For 2D motion the angular momentum operator about the *z*-axis is

$\hat{L}_z = \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x$

- The expectation value of the angular momentum for the stationary coherent state and time-dependent wave packet state which are shown below :

$$\Phi_N(\xi_x, \xi_y; A, \phi) = \frac{1}{\left(\sqrt{1+A^2}\right)^N} \sum_{K=0}^{N} \binom{N}{K}^{1/2} (Ae^{i\phi})^K \tilde{\psi}_{K,\,N-K}(\xi_x, \xi_y)$$

$$\Psi(\xi_x, \xi_y, t) = \sum_{N=0}^{\infty} C_N\, \Phi_N(\xi_x, \xi_y; A, \phi)\, e^{-i(N+1)\omega t}$$

Angular Momentum in 2D Confined Systems

- The position and momentum operators for the harmonic oscillator can be in terms of the creation and annihilation operators.

$$\hat{L}_z = \hat{x}\, p_y - \hat{y}\, p_x$$

$$= i\,\hbar\,\frac{1}{2}\left[\left(\hat{a}_x^{\dagger} + a_x\right)\left(\hat{a}_y - a_y\right) - \left(\hat{a}_y^{\dagger} + a_y\right)\left(\hat{a}_x - a_x\right)\right]$$

$$= i\,\hbar\,\left(\hat{a}_x a_y^{\dagger} - \hat{a}_x\, a_y\right)$$

Angular Momentum in 2D Confined Systems

- The properties of the creation and annihilation operators :

- $\hat{a}_x \hat{a}_y{}^{\dagger}\, \Phi_N(\xi_x, \xi_y; A, \phi)$

$$= \frac{1}{(1+A^2)^{N/2}} \sum_{K=1}^{N} \binom{N}{K}^{1/2} \sqrt{K}\sqrt{N-K+1}\left(Ae^{i\phi}\right)^K \tilde{\psi}_{K-1,N-K+1}(\xi_x, \xi_y)$$

- $\hat{a}_y{}^{\dagger} \hat{a}_x\, \Phi_N(\xi_x, \xi_y; A, \phi)$

$$= \frac{1}{(1+A^2)^{N/2}} \sum_{K=0}^{N-1} \binom{N}{K}^{1/2} \sqrt{K+1}\sqrt{N-K}\left(Ae^{i\phi}\right)^K \tilde{\psi}_{K+1,N-K-1}(\xi_x, \xi_y)$$

Angular Momentum in 2D Confined Systems

- With the orthonormal property of the eignestates :

- $\left\langle \Phi_N(\xi_x,\xi_y;A,\phi)\,|\,\hat{a}_x\hat{a}_y^{\dagger}\,|\,\Phi_N(\xi_x,\xi_y;A,\phi)\right\rangle$

$$=\frac{1}{(1+A^2)^N}\sum_{K=1}^{N}\binom{N}{K-1}^{1/2}\binom{N}{K}^{1/2}\sqrt{K}\sqrt{N-K+1}A^{2K-1}e^{i\phi}$$

$$=\frac{1}{(1+A^2)^N}\sum_{K=1}^{N}\binom{N}{K}K\,A^{2(K-1)}\left(Ae^{i\phi}\right)$$

- $\left\langle \Phi_N(\xi_x,\xi_y;A,\phi)\,|\,\hat{a}_x^{\dagger}\hat{a}_y\,|\,\Phi_N(\xi_x,\xi_y;A,\phi)\right\rangle$

$$=\frac{1}{(1+A^2)^N}\sum_{K=0}^{N-1}\binom{N}{K+1}^{1/2}\binom{N}{K}^{1/2}\sqrt{K+1}\sqrt{N-K}\,A^{2K+1}e^{-i\phi}$$

$$=\frac{1}{(1+A^2)^N}\sum_{K=1}^{N}\binom{N}{K}K\,A^{2(K-1)}\left(Ae^{-i\phi}\right)$$

Angular Momentum in 2D Confined Systems

- Using the property

$$\frac{\partial}{\partial x}(1+x)^N = \frac{\partial}{\partial x}\sum_{K=0}^{N}\binom{N}{K}x^K \quad \Rightarrow \quad N(1+x)^{N-1} = \sum_{K=1}^{N}\binom{N}{K}K\ x^{K-1}$$

- We can obtain $\frac{1}{(1+A^2)^N}\sum_{K=1}^{N}\binom{N}{K}K\,A^{2(K-1)} = \frac{N}{(1+A^2)}$ and

$$\left\langle \Phi_N(\xi_x,\xi_y;A,\phi)\,|\,\hat{L}_z\,|\,\Phi_N(\xi_x,\xi_y;A,\phi)\right\rangle$$

$$= \left\langle \Phi_N(\xi_x,\xi_y;A,\phi)\,|\,i\hbar\left(\partial_x a_y^{?} - \partial_x\ a_y\right)|\,\Phi_N(\xi_x,\xi_y;A,\phi)\right\rangle$$

$$= \frac{i\hbar}{(1+A^2)^N}\sum_{K=1}^{N}\binom{N}{K}K\quad A^{2(K-1)}\left(Ae^{i\phi} - Ae^{-i\phi}\right) = -2N\quad \hbar\frac{A}{1+A^2}\sin\phi$$

Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits

- The time-independent Schrödinger equation for a 2D harmonic oscillator with commensurate frequencies can generally given by

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{2}m(\omega_x^2 x^2+\omega_y^2 y^2)\right]\psi(x,y)=E\,\psi(x,y)$$

$$\omega_x = q\omega \qquad \omega_y = p\omega$$

ω is the common factor of the frequencies by ω_x and ω_y, and *p* and *q* are relative prime integers

- The eigenfunction and the eigenvalue of the 2D harmonic oscillator with commensurate frequencies are given by

$$\tilde{\psi}_{m,n}(\xi_x,\xi_y)=\left(2^{n+m}\,m!\,n!\cdot\pi\right)^{-1/2}\,e^{-(\xi_x^2+\xi_y^2)/2}H_m(\xi_x)H_n(\xi_y)$$

$$E_{m,n}=\left(m+\frac{1}{2}\right)\hbar\omega_x+\left(n+\frac{1}{2}\right)\hbar\omega_y \qquad \xi_x=\sqrt{m\omega_x/\hbar}\,x \qquad \xi_y=\sqrt{m\omega_y/\hbar}\,y$$

Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits

- The eigenfunction is separable, so the corresponding Schrödinger coherent state can be expressed as the product of two 1D coherent states:

$$\Psi(\xi_x,\xi_y,t)=\left(\sum_{m=0}^{\infty}\frac{(\alpha_x e^{i\phi_x})^n}{\sqrt{m!}}e^{-\alpha_x^2/2}\frac{1}{\sqrt{2^m m!\sqrt{\pi}}}H_m(\xi_x)e^{-\xi_x^2/2}e^{-i(m+1/2)q\omega t}\right)$$
$$\times\left(\sum_{n=0}^{\infty}\frac{(\alpha_y e^{i\phi_y})^n}{\sqrt{n!}}e^{-\alpha_y^2/2}\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}H_n(\xi_y)e^{-\xi_y^2/2}e^{-i(n+1/2)p\omega t}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_x e^{i\phi_x})^m(\alpha_y e^{i\phi_y})^n}{\sqrt{m!}\sqrt{n!}}e^{-(\alpha_x^2+\alpha_y^2)/2}\,\tilde{\psi}_{m,n}(\xi_x,\xi_y)\,e^{-i(qm+pn+q/2+p/2)\omega t}$$

Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits

- It is clear that the center of the wave packet follows the motion of a classical 2D isotropic harmonic oscillator, i.e.,

$$\xi_x = \sqrt{2}\alpha_x \cos(q\omega t - \phi_x)\,; \quad \xi_y = \sqrt{2}\alpha_y \cos(p\omega t - \phi_y)$$

- The set of states with indices (m,n) in last page can be divided into subsets characterized by a pair of indices (u_x,u_y) given by $m \equiv u_x \pmod p$ and $n \equiv u_y \pmod q$
- Schrödinger coherent state can be rewritten as

$$\Psi(\xi_x,\xi_y,t) = \left(\sum_{u_y=0}^{q-1} \sum_{u_x=0}^{p-1} \sum_{N_y=0}^{\infty} \sum_{N_x=0}^{\infty} \frac{(\alpha_x\ e^{i\phi_x})^{pN_x+u_x} (\alpha_y\ e^{i\phi_y})^{qN_y+u_y}}{\sqrt{(pN_x+u_x)\ !}\ \sqrt{(qN_y+u_y)\ !}} e^{-(\alpha_x^2+\alpha_y^2)/2} \right.$$

$$\left. \times \tilde{\psi}_{pN_x+u_x,qN_y+u_y}(\xi_x,\xi_y)\ e^{-i[pq(N_x+N_y)+q(u_x+1/2)+p(u_y+1/2)]\omega t} \right)$$

Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits

- The 2D Schrödinger coherent state is divided into a product of two infinite series and two finite series
- With the representation of the Cauchy product, the terms $\tilde{\psi}_{pN_x+u_x,qN_y+u_y}(\xi_x,\xi_y)$ can be arranged diagonally by grouping together those terms for which

$N_x + N_y = N$:

$$\Psi(\xi_x,\xi_y,t) = \left(\sum_{u_y=0}^{q-1} \sum_{u_x=0}^{p-1} \sum_{N=0}^{\infty} \sum_{K=0}^{N} \frac{(\alpha_x e^{i\phi_x})^{pK+u_x} (\alpha_y e^{i\phi_y})^{q(N-K)+u_y}}{\sqrt{(pK+u_x)!}\,\sqrt{[q(N-K)+u_y]!}} e^{-(\alpha_x^2+\alpha_y^2)/2} \right.$$
$$\left. \times \tilde{\psi}_{pK+u_x,\,q(N-K)+u_y}(\xi_x,\xi_y)\, e^{-i[pqN+q(u_x+1/2)+p(u_y+1/2)]\omega t} \right)$$

$$= \sum_{u_y=0}^{q-1} \sum_{u_x=0}^{p-1} \sum_{N=0}^{\infty} e^{-(\alpha_x^2+\alpha_y^2)/2} (\alpha_x e^{i\phi_x})^{u_x} (\alpha_y e^{i\phi_y})^{qN+u_y} e^{-i[pqN+q(u_x+1/2)+p(u_y+1/2)]\omega t}$$
$$\times \left\{ \sum_{K=0}^{N} \frac{(\alpha_x^p/\alpha_y^q)^K [e^{i(p\phi_x-q\phi_y)}]^K}{\sqrt{(pK+u_x)!}\,\sqrt{[q(N-K)+u_y]!}} \tilde{\psi}_{pK+u_x,\,q(N-K)+u_y}(\xi_x,\xi_y) \right\}$$

Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits

- These stationary coherent states are physically expected to be associated with the Lissajous trajectories.
- The minor indices u_x and u_y essentially do not affect the characteristics of the stationary states.
- Including the normalization condition, the stationary coherent states in Cartesian coordinates are given by

$$\Phi_{N,u_x,u_y}(\xi_x,\xi_y;A,\phi)=\left(\sum_{K=0}^{N}\frac{A^{2K}}{(pK)!\cdot[q(N-K)+u_y]!}\right)^{-1/2}$$

$$\times\sum_{K=0}^{N}\frac{[Ae^{i\phi}]^K}{\sqrt{(pK)!}\sqrt{[q(N-K)+u_y]!}}\tilde{\psi}_{pK+u_x,\,q(N-K)+u_y}(\xi_x,\xi_y)$$

$$A=\frac{(\alpha_x)^p}{(\alpha_y)^q},\qquad \phi=p\phi_x-q\phi_y$$

Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits

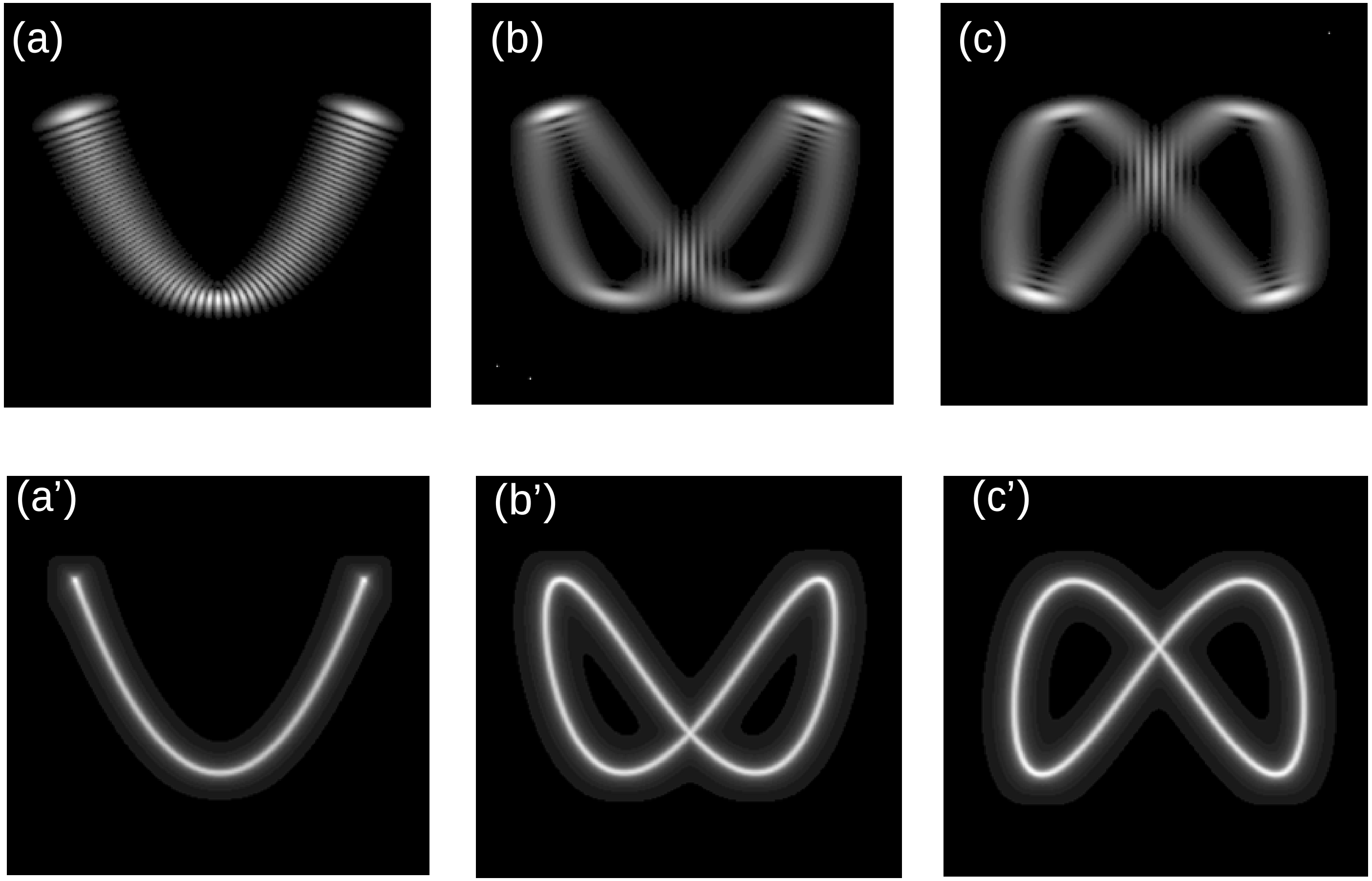
- The stationary coherent states associated with the Lissajous trajectories are the superposition of degenerate eigenstates with the relative amplitude factor *A* and phase factor ϕ .
- The relative amplitude factor *A* and phase factor ϕ in the stationary coherent states $\Phi_{N,u_x,u_y}(\xi_x,\xi_y;A,\phi)$ are explicitly related to the classical variables $(\alpha_x,\alpha_y,\phi_x,\phi_y)$
- the eigenenergies of the stationary coherent states $\Phi_{N,u_x,u_y}(\xi_x,\xi_y;A,\phi)$ are found to be

$$E_{N,u_x,u_y} = \left[pqN + q(u_x + 1/2) + p(u_y + 1/2)\right]\hbar\omega$$

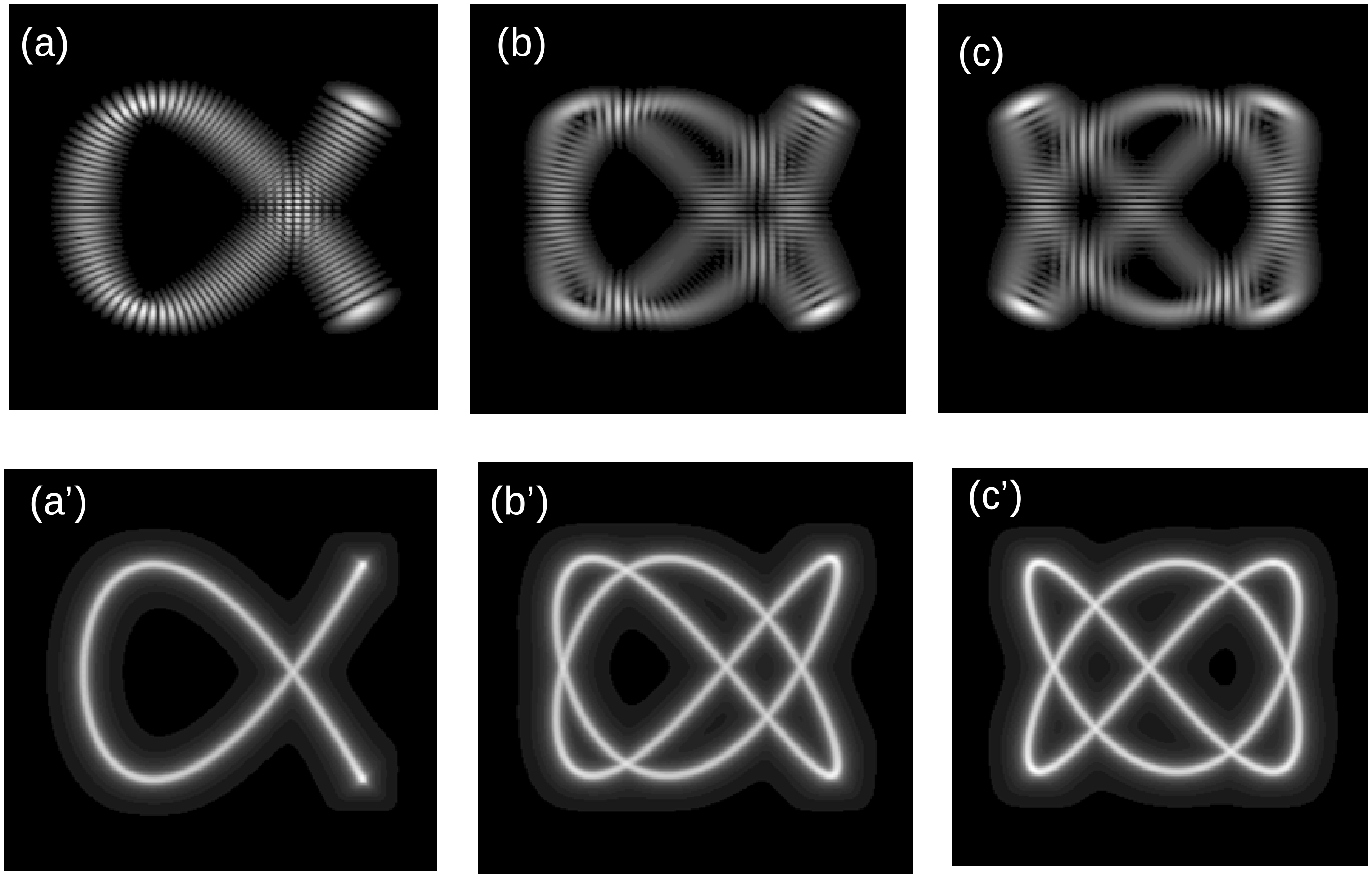
Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits

- Next three figures depict the comparison between the quantum wave patterns $|\Phi_{N,0,0}(\xi_x,\xi_y;A,\phi)|^2$ and the corresponding classical periodic orbits for $p:q$ to be $2:1$, $3:2$, and $4:3$, respectively.
- Three different phase factors, $\phi=0$, $\phi=0.3\pi$, and $\phi=0.6\pi$, are displayed in each figure.
- The behavior of the quantum wave patterns in all cases can be found to be in precise agreement with the classical Lissajous figures.

Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits



Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits



Quantum Stationary Coherent States for Classical Lissajous Periodic Orbits