Lab 2: Chaos

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Abstract

The chaotic behaviour of a driven pendulum is explored. Phase space behaviours of a random response, non-chaotic periodic response, and chaotic response are generated from computational models. The functional form of each is compared, demonstrating the strange attractor as a distinguishing feature of a chaotic response. The motion of a physical pendulum is observed under damped and damped-driven conditions. The motion is characterized according to its phase space output, and determined to be chaotic.

1 Chaos

Chaos is observed in many non-linear physical systems. It is the condition that a system's outcome is strongly sensitive to initial conditions. The changing conditions as the system evolves affect the outcome such that predicting the future state becomes impossible. The motion of a driven pendulum, for example, becomes unpredictable as the driving frequency and natural frequency of the pendulum interact. Damping can constrain the motion, and we find that while the motion is unpredictable, it still displays certain characteristics that can be analyzed.

For a finite period of time, chaotic behaviour isn't completely discernable from periodic behaviour, because the possibility exists that the function may repeat itself at some future time. To identify chaos, one makes a judgment after enough time has elapsed to assume for practical purposes the function will not repeat. Distinctions are seen between a random variable, periodic variable, and chaotic variable in phase space and poincar sections. [3]

2 Phase Space and Poincar Sections

A 2-dimensional phase space is a useful environment in which to identify the chaos in the motion in a coordinate. For a pendulum, the convenient coordinate is its angular position θ, with the angular velocity response θ'. In figure 1, observe how a periodic variable can be identified in phase space.

Such a function should display significant variation and interceptions of its apex in phase space. [2]

Chaotic Attractor

Chaotic attractors can be observed when forcing interacts with damping in a system. A damped system that is not forced has critical points where the velocity converges to zero about some position. Under forced conditions, the velocity does not convergse to zero, but the motion produces orbits about these critical points in phase space, and we call these chaotic attractors. These attractors are a primary identifying characteristic of chaos, and should be observable in the chaotic motion of the forced pendulum. [1]

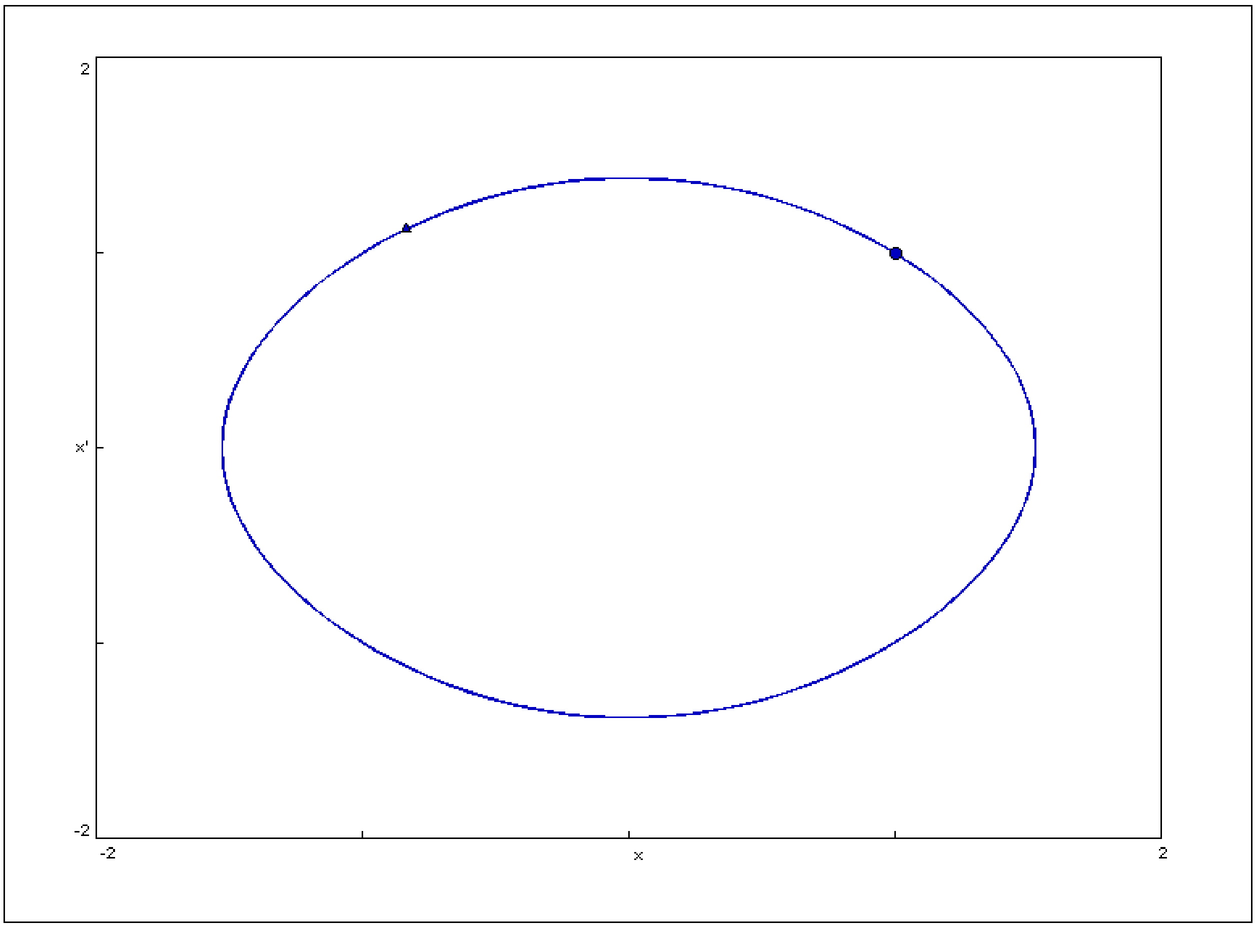

Figure 1: Computer-generated model of a periodic function in phase space. A pendulum's angular coordinate would correspond to $\theta = x$. The motion is predictable and orbits a single point.

3 Modeling Chaos in a Driven Pendulum

In the angular coordinate θ, the equation of motion of a driven simple pendulum is

$$\frac{d^2}{dt^2} = \frac{{\omega_0}^2}{I} sin(\theta) - \frac{\alpha}{I}\frac{d\theta}{dt} + \frac{f}{I} cos(\omega t + \phi).$$

Here, ω_0 represents the natural frequency of the pedulum, also its resonant frequency. The system will respond most strongly to the driver at this frequency. α is a dampening term – this can take a variety of forms, and in the experiment of section **??** is produced by a neodymium magnet interacting with the metal wheel of the pendulum. f is the forcing amplitude where ω is the forcing frequency, offset from the angular coordinate by a phase ϕ. I is the moment of inertia of the pendulum.

If there is no damping, observing the pendulum's natural frequency interacts with the driving frequency is straight-forward, for example in figure 2. The path is traced from the circle to the triangle. The natural (un-driven) response of the pendulum is seen in the tall cirular strokes, which represent the pendulum's weight attempting to bring the pendulum to equilibrium. The driving frequency produces small-amplitude variations when the pendulum has a high angular speed and when the pendulum has a low angular speed, it can easily reversed the motion of the pendulum. This sensitivity is characteristic of chaotic motion.

The path through phase space is apparently unstable and could easily take off toward either extreme of the θ coordinate. The large swoops the

4 Chaos compared against Randomsdfness

A random variable has no discernable pattern in its output. If a variable has a random response as a function of time, then it will appear as a random scatter in both the time domain and in phase space. A random response to the position coordinate is plotted in figure **??**, in phase space. The random character can be see in the fact that the distribution of first-derivative responses has no discernable pattern.

A poincar map takes a periodic input and samples the output of a variable response in phase space. If the system is periodic with that sampling period, the poincar map will plot only a single dot, which is the same response seen after each period, e.g. figure **??**. If the response is random, the poincar map will have a random distribution, similar to that seen in the phase map (figure **??**. If the response is chaotic, we expect to see a distribution of points that is not entirely random but is not confined to a single point. This further demonstrates the continuum behaviour where a chaotic variable extends the behaviour of a periodic variable but not to the point of randomness.

The driven pendulum is a physical system that is known to exhibit chaotic behaviour. The model for this system can be expressed analytically given a small-angle approximation

The phase map and poincar map are generated from the model of a driven pendulum

5 Discussion

References

[1] G.L. Baker and J.P. Gollub. *Chaotic dynamics: an introduction*. 2nd ed. The Pitt Building, Trumpington Street, Cambridge CB2 1RP: Cambridge University Press, 1996. ISBN: 0521471060.

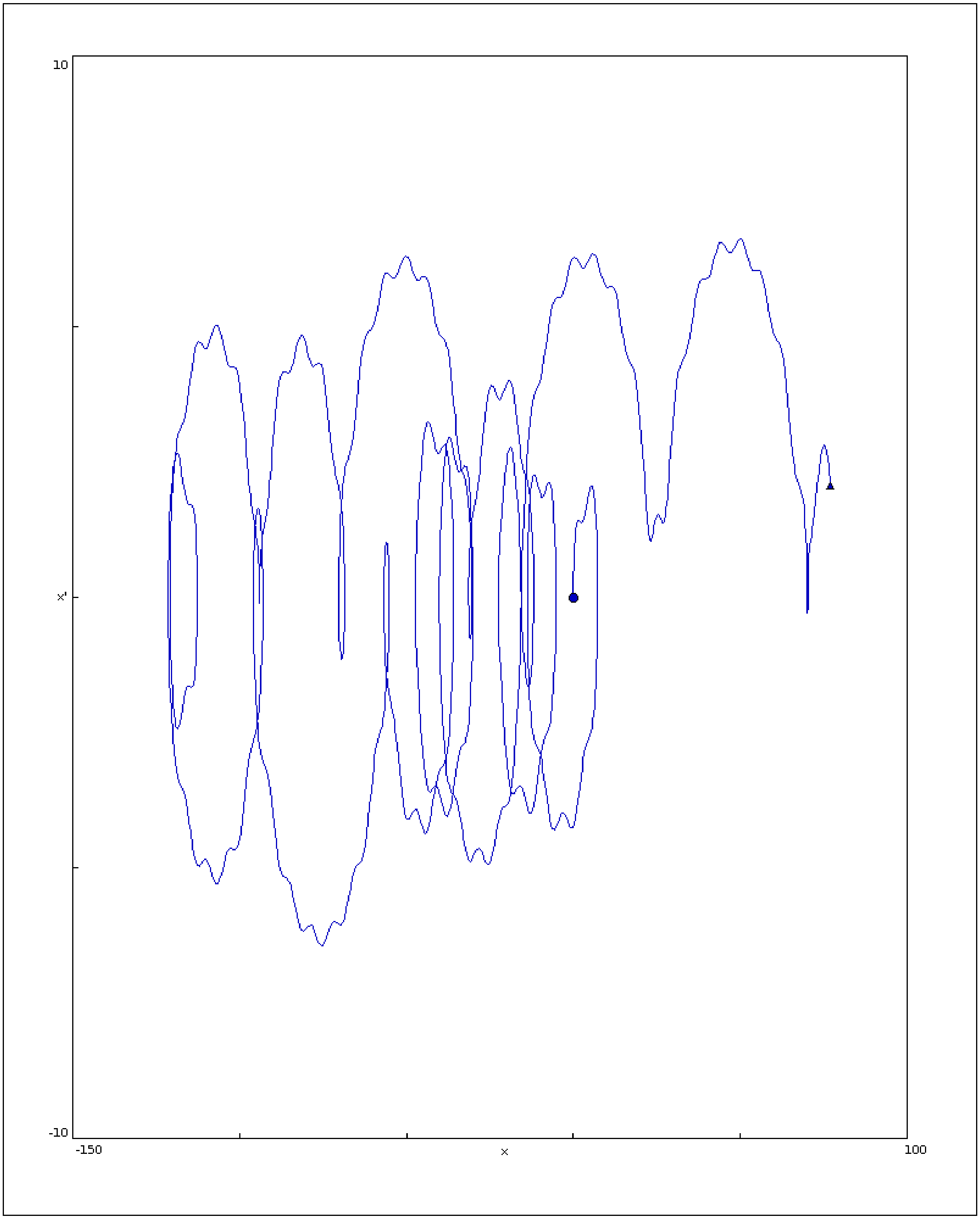

Figure 2: Computer-generated model of a driven pendulum with no damping. Arguments: $\theta = x, f = 1, I = 1, \omega = 0.5, \alpha = 0, k = \frac{\omega_0^2}{I} = 1.5, \phi = 0$, 180 time steps.

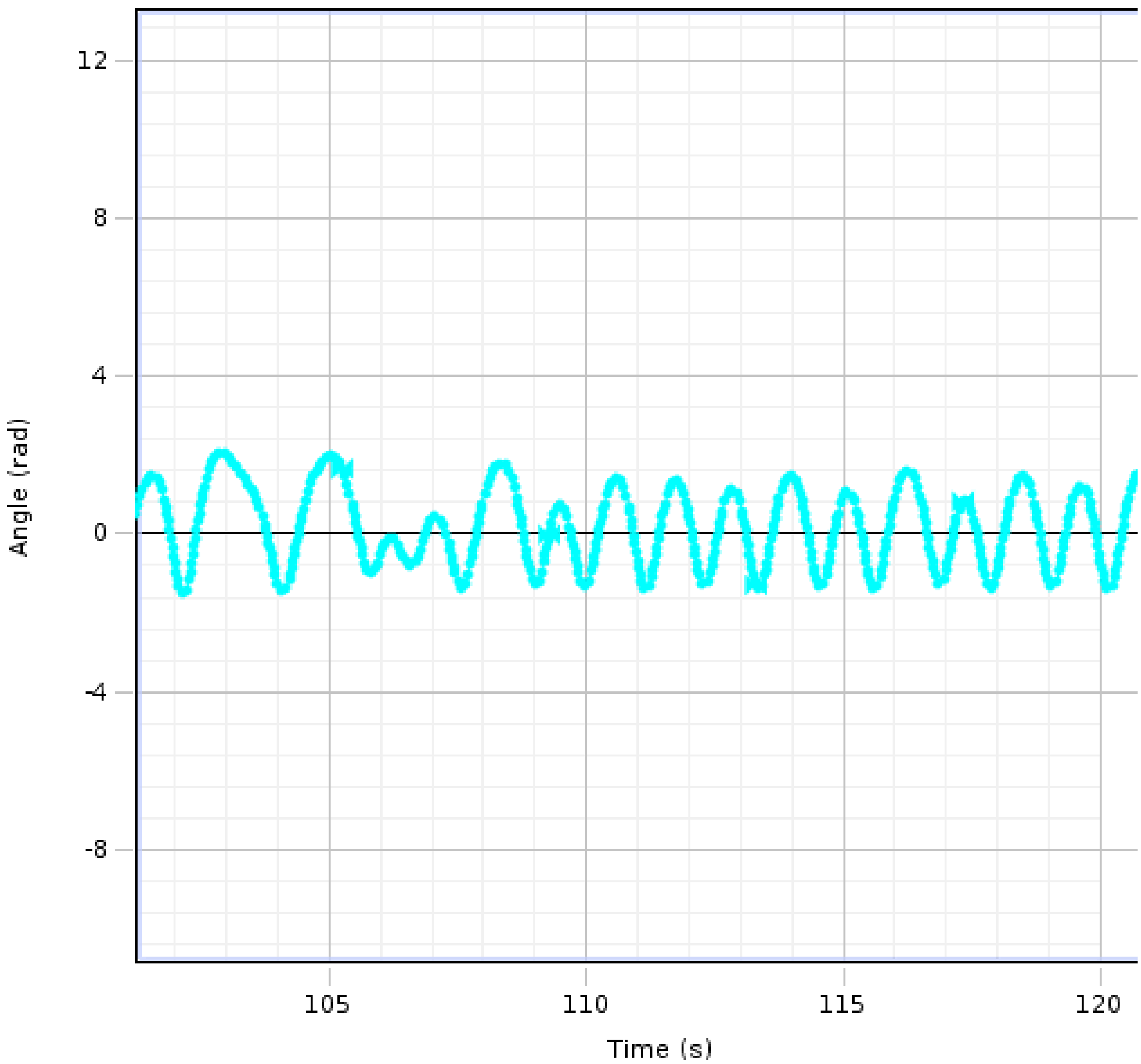

Figure 3: Computer-generated model of a driven pendulum with no damping. Arguments: $\theta = x, f = 1, I = 1, \omega = 0.5, \alpha = 0, k = \frac{\omega_0^2}{I} = 1.5, \phi = 0$, 180 time steps.

[2] Adilson E. Motter and David K. Campbell. “The tangled tale of phase space”. In: *Physics Today* (May 2013), pp. 27–32.

[3] David D. Nolte. “The tangled tale of phase space”. In: *Physics Today* (Apr. 2010), pp. 33–38.