Lab 2: Chaos in a Driven Pendulum

Otho Ulrich, Eugene Kopf

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Abstract

The chaotic behaviour of a driven pendulum is explored. Phase space behaviours of a non-chaotic periodic response and chaotic response are generated from computational models. The form of each is compared, demonstrating the chaotic attractor as a distinguishing feature of chaos. The motion of a physical pendulum is observed under damped and damped-driven conditions. The motion is explored according to its phase space output, and determined to be chaotic.

1 Chaos

Chaos is observed in many non-linear physical systems. It is the condition that a system's outcome is strongly sensitive to initial conditions. The changing conditions as the system evolves affect the outcome such that predicting the future state becomes impossible. The motion of a driven oscillator, such as a driven pendulum, for example, becomes unpredictable as the driving frequency and natural frequency of the pendulum interact. Damping can constrain the motion, and we find that while the motion is unpredictable, it still displays certain characteristics that can be analyzed.

For a finite period of time, chaotic behaviour isn't completely discernable from periodic behaviour, because the possibility exists that the function may repeat itself at some future time. To identify chaos, one makes a judgment after enough time has elapsed to assume for practical purposes the function will not repeat. Distinctions are seen between a periodic variable and a chaotic variable in phase space and poincaré sections. [5]

1.1 Model of a Driven Pendulum

The driven pendulum exemplifies chaotic motion. In the angular coordinate θ, the equation of motion of a driven pendulum is

$$\frac{d^2\theta}{dt^2} = \frac{{\omega_0}^2}{I} sin(\theta) - \frac{\alpha}{I}\frac{d\theta}{dt} + \frac{f}{I} cos(\omega t + \phi).$$

Here, ω_0 represents the natural frequency of the pedulum, also its resonant frequency. The system will respond most strongly to the driver at this frequency. α is a damping term – this can take a variety of forms, and in the experiment of section 3 is produced by a neodymium magnet interacting with the metal wheel of the pendulum. f is the forcing amplitude where ω is the forcing frequency, offset from the angular coordinate by a phase ϕ. I is the moment of inertia of the pendulum.

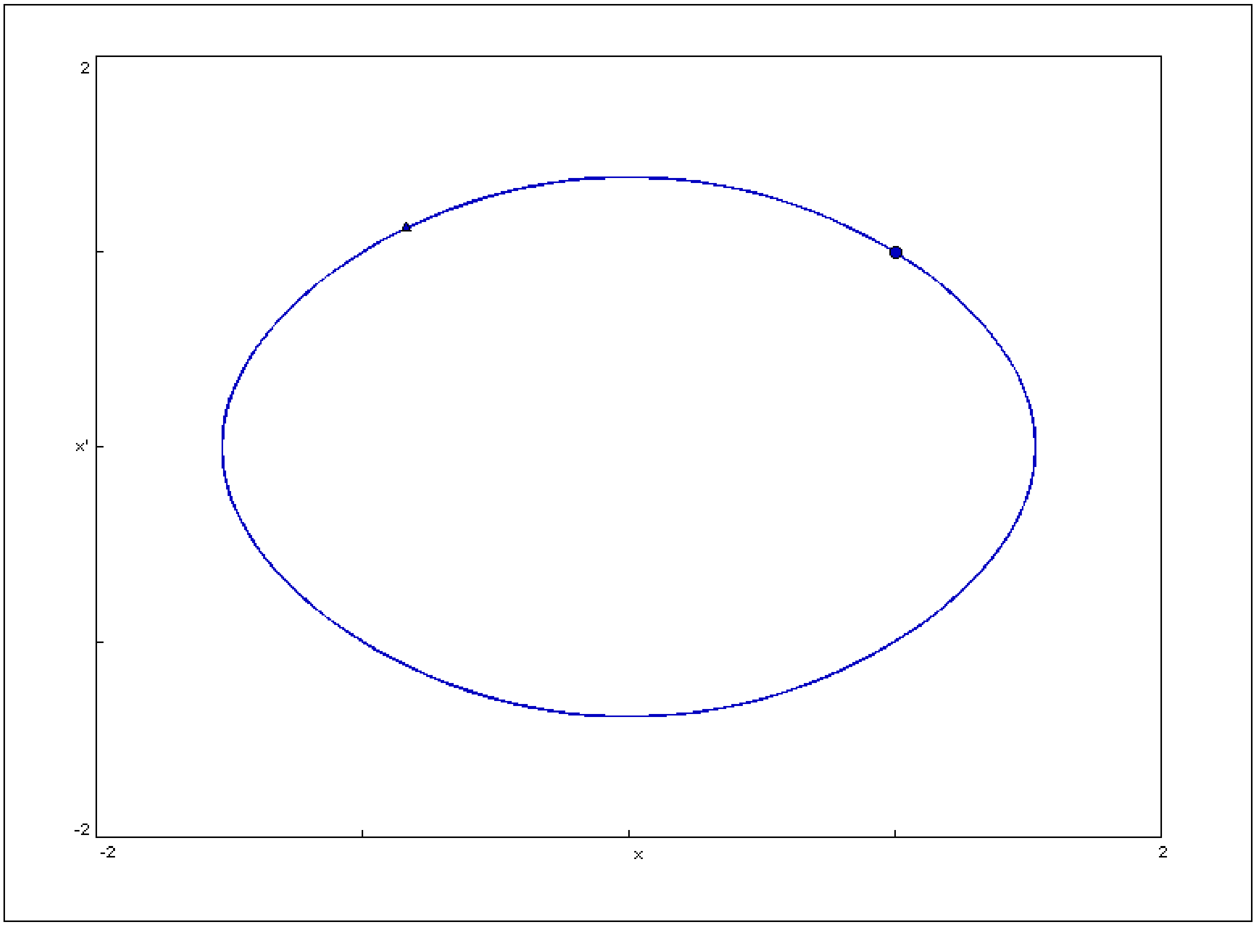


Figure 1: Computer-generated model of a periodic function in phase space. A pendulum's angular coordinate would correspond to $\theta = x$. The motion is predictable and orbits a single point.

2 Phase Space

A 2-dimensional phase space is a useful environment in which to identify the chaos in the motion in a coordinate. For an oscillator, the convenient coordinate is its angular position θ, with the angular velocity response θ'. In figure 1, observe how a periodic variable can be identified in phase space.

2.1 Chaotic Attractor

Figure 2 demonstrates a damped oscillator, which exhibits stable critical points where the angular velocity goes to zero. Under forced conditions, the angular velocity does not converge to zero, but the motion produces orbits about these critical points in phase space, and we call these points chaotic attractors. Poincaré first postulated that chaos would be exemplified by complicated paths that roughly follow one of these orbits (the apex) about these attractors. [4] Attractors are a primary identifying characteristic of chaos, and should be observable in the chaotic motion of a forced pendulum. [1] A driven oscillator's path orbits around these critical points but can be seen to jump between them in an unpredictable way along the position coordinate; observe figure 3.

Damping still plays an important role in this chaotic motion; figure 4 shows a driven pendulum with no damping, where the motion freely exhibits the effects of both the natural and forcing frequency, and becomes extremely complicated. The natural (un-driven) response of the pendulum

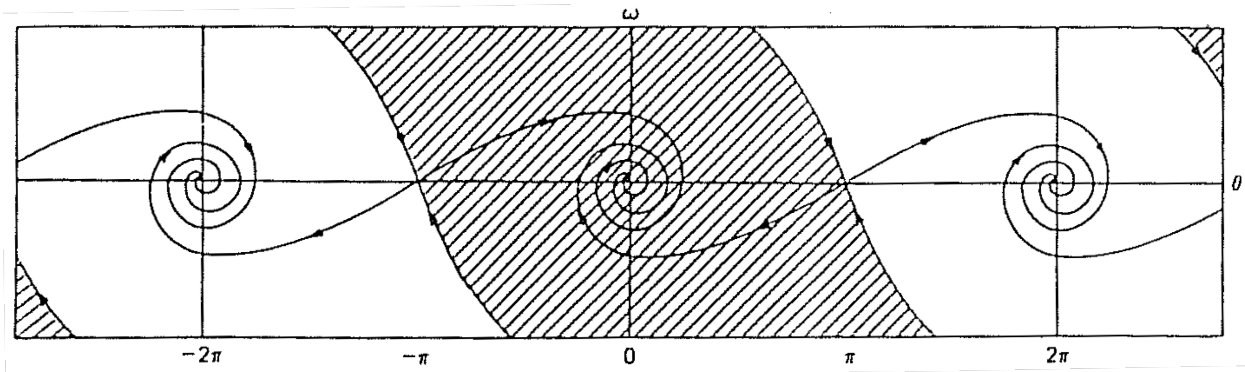


Figure 2: Model of a damped oscillator in phase space. The critical points are stable, since the velocity approaches zero from all points within the associated region. [1]

is seen in the tall cirular strokes, which represent the pendulum's weight attempting to bring the pendulum to equilibrium. The driving frequency produces small-amplitude variations when the pendulum has a high angular speed and when the pendulum has a low angular speed, it can easily reverse the motion of the pendulum. The oscillator jumps around to many possible states, making it difficult to discern attractors. Chaotic attractors are much more clear when forcing interacts with damping in a system; figure 3 shows this very well.

Computer models were generated using the Chaos for Java program written by Brian Davies. [3] The path is computed from the circle to the triangle.

2.2 Poincaré Map

A Poincaré section is a 2D phase space cross-section; in the case of the driven pendulum, the cross-section at a phase ϕ from the forcing term. One draws a map between each successive point to create a Poincaré map, which is a useful representation of a system's behaviour in phase space. A poincaré section of the driven pendulum model is shown in figure 5. An undriven and undamped oscillator will always return to the same point after one period, so a Poincaré map sampled using the period corresponding to the oscillator's natural frequency will consist of a single dot. The existence of multiple points at the same phase indicates chaotic motion.

3 Experimental Driven Pendulum

An experimental driven pendulum is built in order to determine whether its motion is chaotic. The apparatus is shown in figure 6. The springs are connected so that they both hold about equal tension while the pendulum is at its top-most position, and so that they can stretch over at least one full turn of the pendulum in either direction. The tension was not measured precisely, nor the weight of the pendulum or the damping of the pendulum. The amplitude of the driving arm was 1.45" $\pm$ 0.1". A photogate is used to determine the driving period when necessary.

3.1 Resonant Frequency

To determine the natural frequency of the pendulum, it was released from the top position under damped conditions and its frequency of oscillation was measured. The waveform is shown in figure 7. Fourier transformations of the waveform in figure 7 reveal the natural frequency at where the

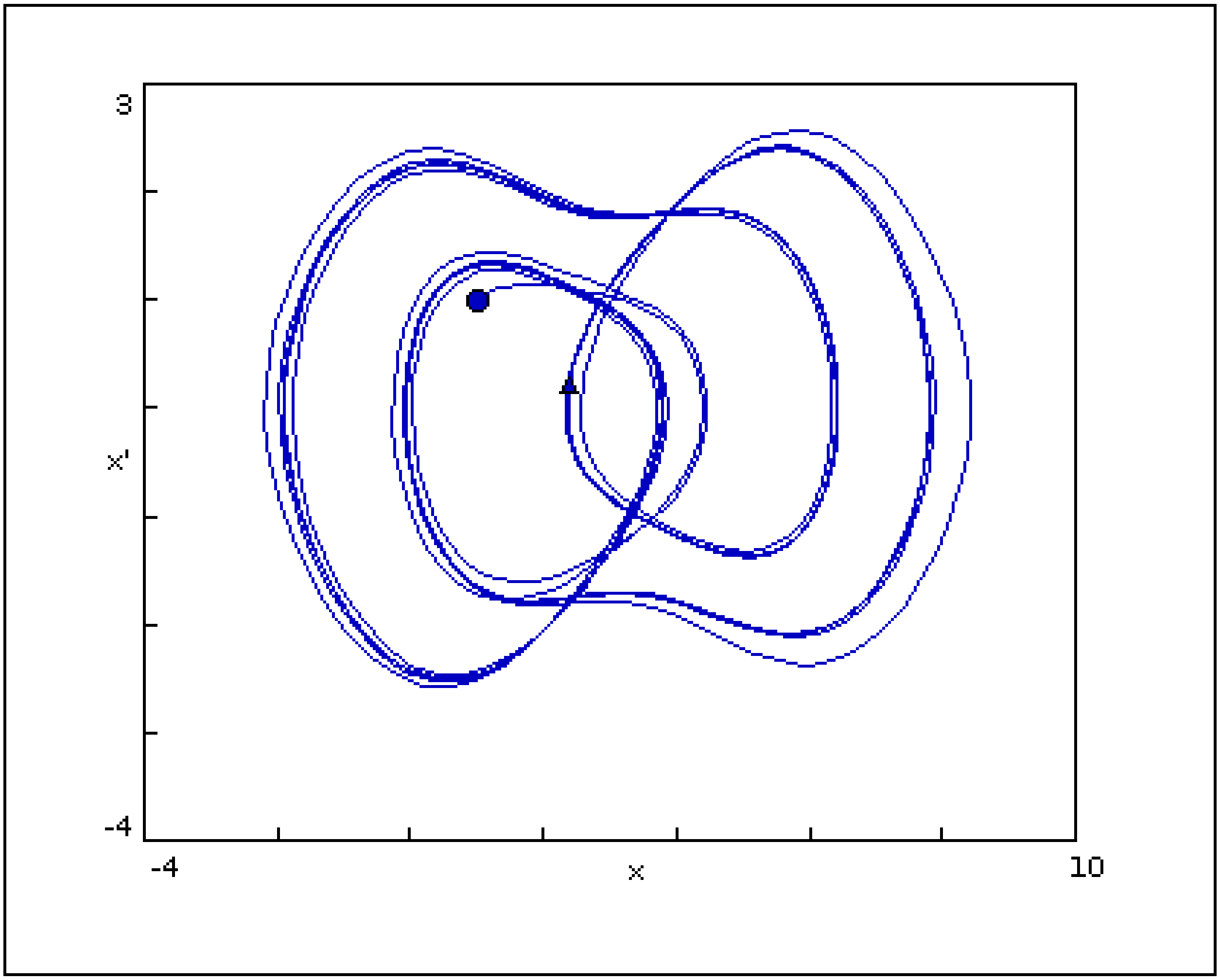


Figure 3: Computer-generated model of a damped and driven pendulum's angular motion. The motion jumps between critical points (called attractors in this context), and exhibits a semi-stable low-energy state around one of attractors. Arguments: $\theta = x, f = 1, I = 1, \omega = 2/3, \alpha = 0.7, k = \frac{\omega_0^2}{I} = 2, \phi = 0, \theta(0) = 1, \theta'(0) = 1$, 150 time steps.

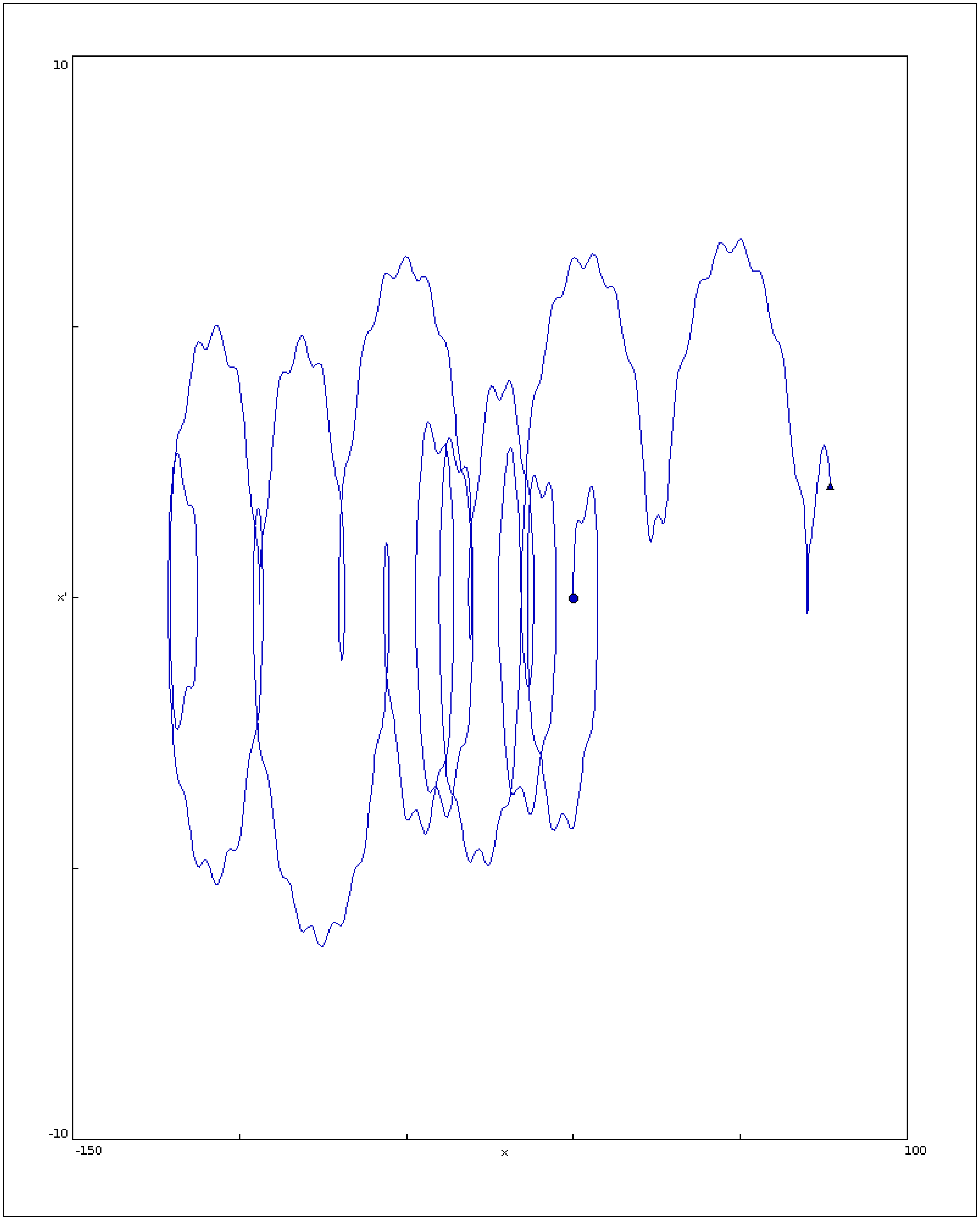


Figure 4: Computer-generated model of a driven pendulum with no damping. The path through phase space is unstable and could easily take off toward either extreme of the θ coordinate. This sensitivity is characteristic of chaotic motion. Arguments: $\theta = x, f = 1, I = 1, \omega = 0.5, \alpha = 0, k = 1.5, \phi = 0$, 180 time steps.

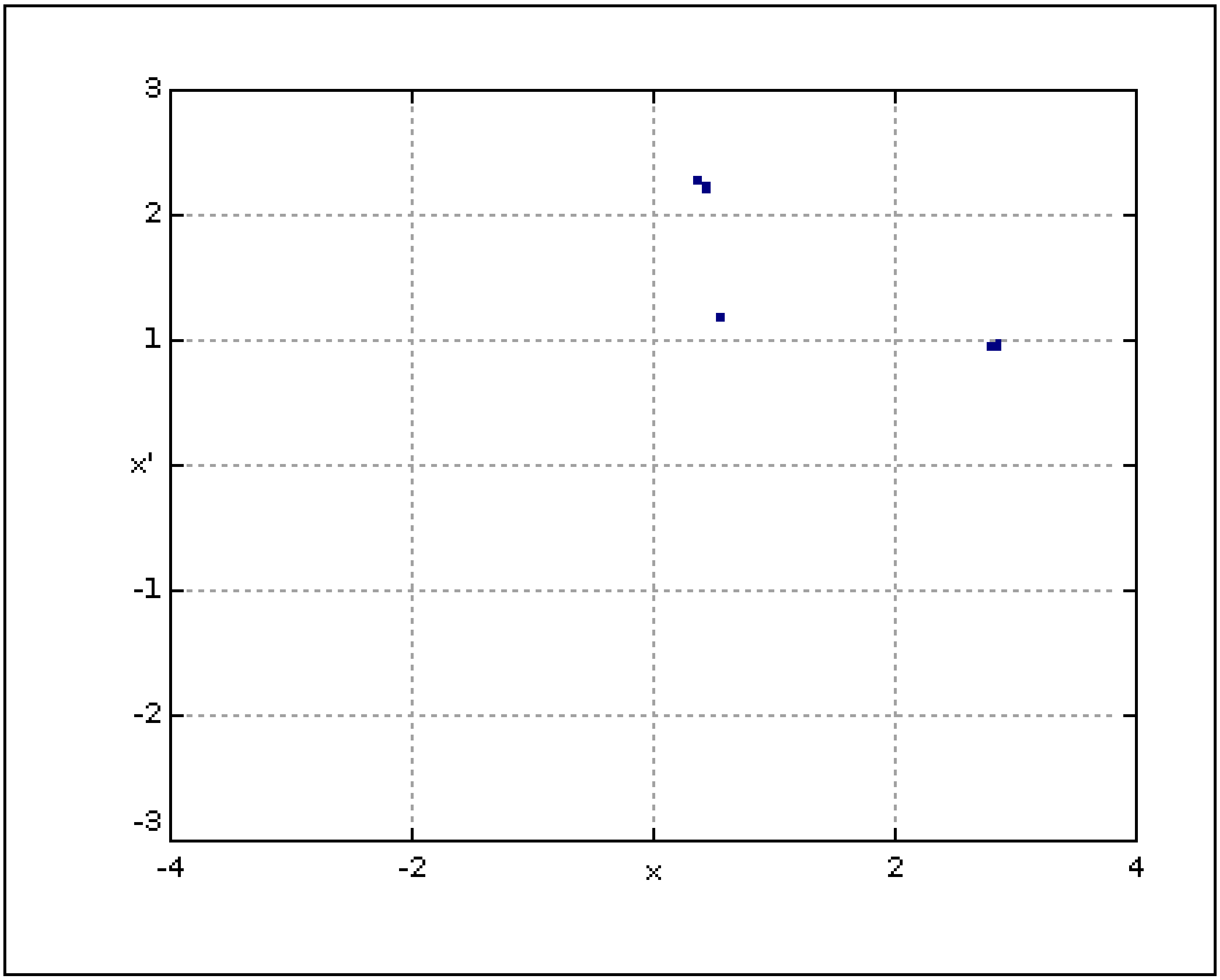


Figure 5: Poincaré section of the damped and driven pendulum from figure 3 at $\phi = 0$. Several dots are observed at this sampling phase, so the motion can be deemed chaotic.

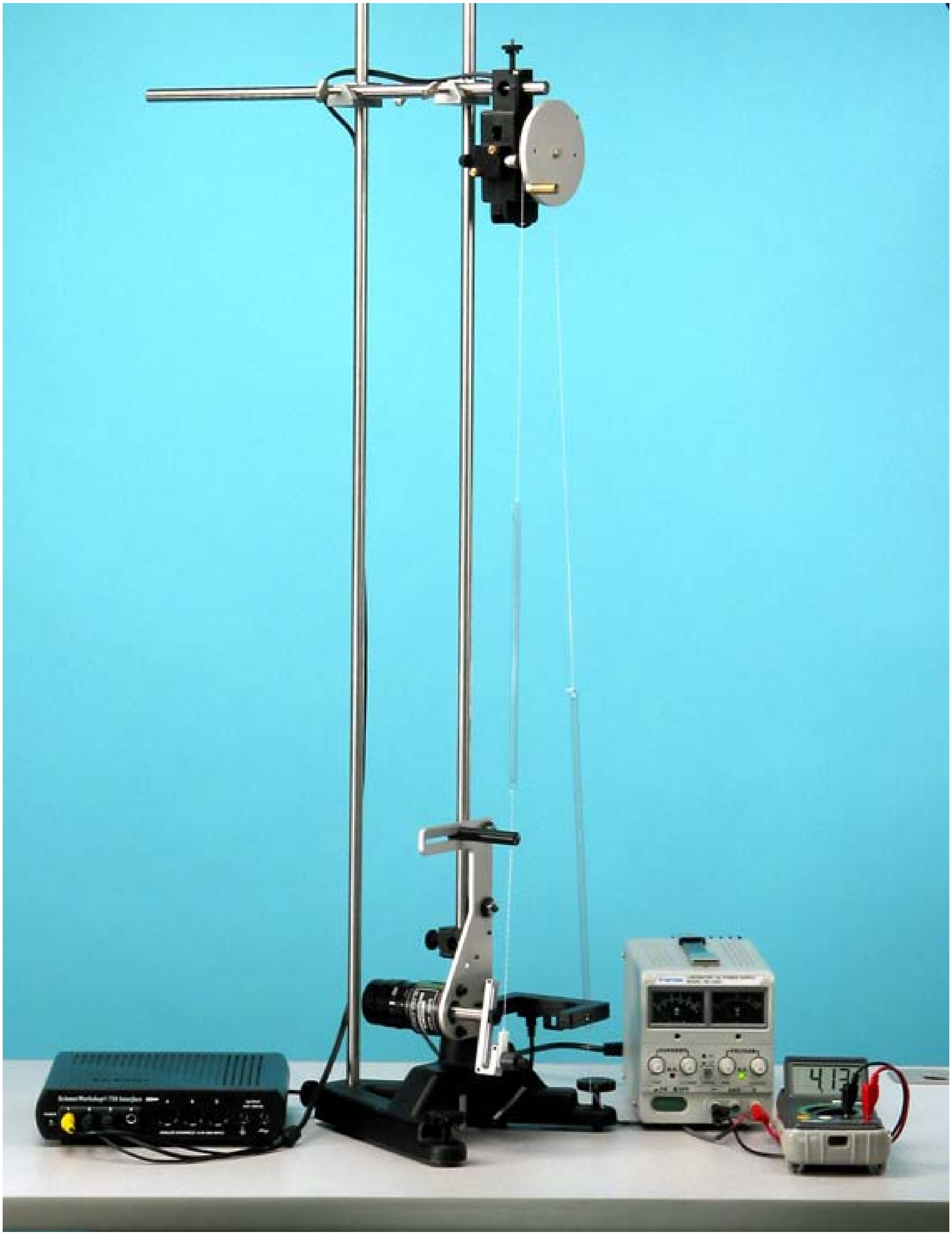


Figure 6: Experimental driven pendulum. This angular motion of the round metal pendulum at the top is expected to exhibit chaotic motion. [2]

power spectral density peaks, discounting the low-frequency peak from the flattened waveform due to damping. The natural frequency is estimated to be $0.93 \pm 0.6Hz$.

3.2 Periodic Motion

To produce periodic motion, the driving arm is run with a driving period $1.45 \pm 0.02s$. The observed motion is plotted in figure 8. The motion still suggests the possibility of chaos by the variation in the path taken about the critical point, but the poincaré section in figure 8 indicates that the orbit is likely converging. The period of the observed motion is consistent with the driving period.

3.3 Chaotic Motion

The driving arm period was increased until the pendulum was exhibiting visibly complex behaviour. The final period was $1.13 \pm 0.02s$. In figure 9 the observed motions are plotted. Two chaotic attractors are strikingly visible in the phase diagram, and the poincaré plot shows significant deviation through the phase.

The natural frequency of the pendulum is tested next. The driving period corresponding to the natural frequency was chosen as $1.26 \pm 0.02s$. This driving frequency found a stronger balance between the two attractors observed at the previous frequency. Figure 10 shows plots of the motion. While performing this test, it became clear that the driving arm was near the resonance frequency, as the apparatus began to shake itself to tipping.

One final driving period was chosen for good measure. At $1.15 \pm 0.02s$, the motion appears closer to sinsuisoidal than in the pervious tests. Nonetheless, the attractors can be observed in the phase diagram and the paths through the poincaré section are limited to the same integrated area.

4 Discussion

The periodic motion is observable in the first experimental case, plotted in figure 8. Once the driving arm period was increased from that frequency to one that induced chaotic motion, the predicted spread of intercepts through the poincaré section and the orbits about the chaotic attractors were visually confirmed. It would have been useful to produce the phase diagram of the damped, undriven pendulum to compare the locations of those critical points to the chaotic attractors. However, as Poincaré predicted so long ago, it is quite evident that constrained chaotic motion does follow complex paths about attractors, and this motion is evident in the observed motion of the damped, driven pendulum.

References

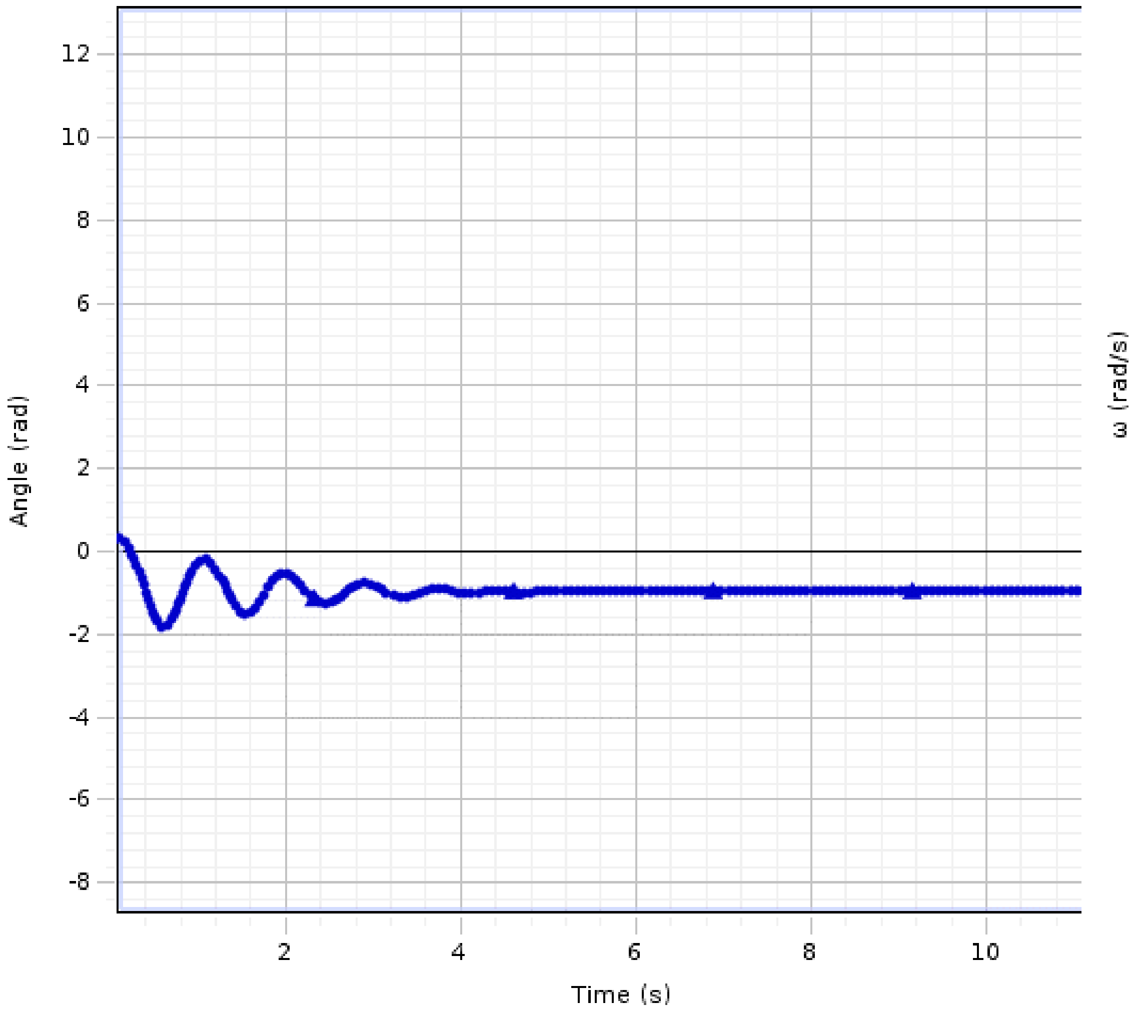
[1] G.L. Baker and J.P. Gollub. *Chaotic dynamics: an introduction*. 2nd ed. The Pitt Building, Trumpington Street, Cambridge CB2 1RP: Cambridge University Press, 1996. ISBN: 0521471060.

[2] Clement Burns. *Chaos Lab V3*. Lab Guide. Western Michigan University, 2017.

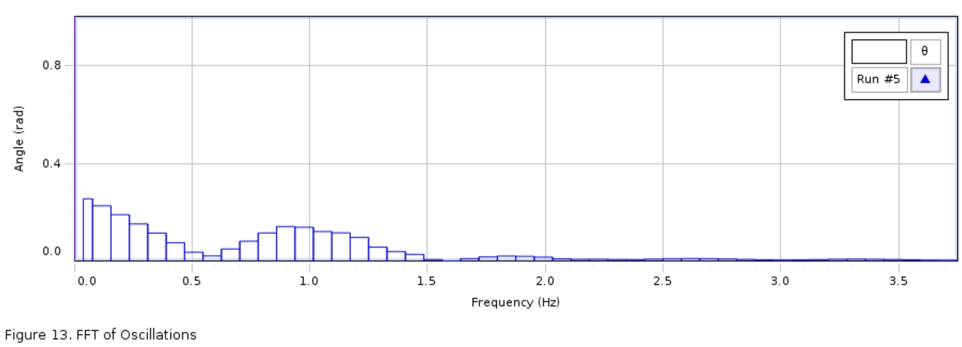
[3] Brian Davies. *Chaos for Java*. Program. Exploring Chaos, 2017.

[4] Adilson E. Motter and David K. Campbell. "The tangled tale of phase space". In: *Physics Today* (May 2013), pp. 27–32.

[5] David D. Nolte. "The tangled tale of phase space". In: *Physics Today* (Apr. 2010), pp. 33–38.

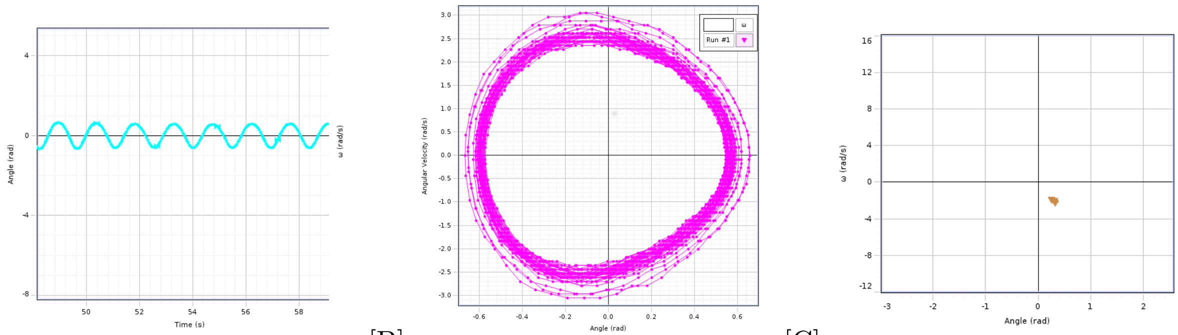


[A]



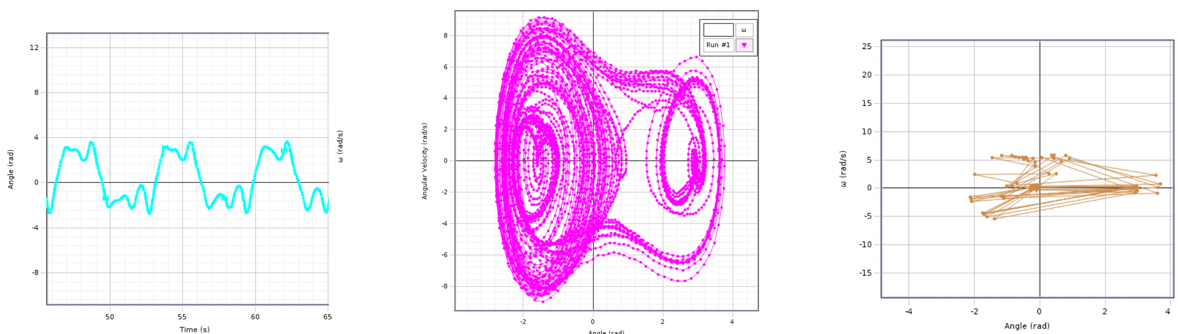
[B]

Figure 7: [A] The damped oscillations of the experimental pendulum. The fourier transforms of this function reveal the natural frequency of the pendulum, in figure B. [B] The fourier frequency-space representation of the damped pendulum. The peak near $0.9Hz$ is the natural/resonant frequency of this pendulum.



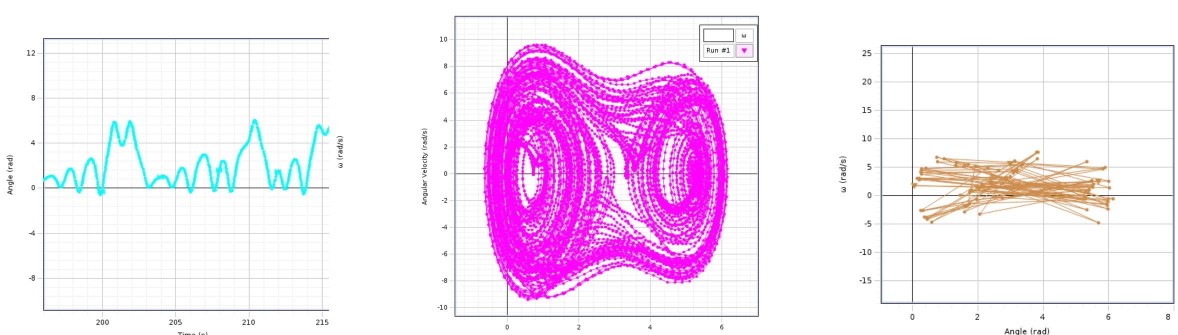
[A] [B] [C]

Figure 8: [A] The observed motion when the driving arm is run with period $1.45 \pm 0.02s$ is periodic at $1.48 \pm 0.08s$. [B] The periodic nature is recognizable in phase space. [C] The poincaré section shows the orbit is likely converging.



[A] [B] [C]

Figure 9: [A] The observed motion when the driving arm is run with period $1.13 \pm 0.02s$ is chaotic. [B] Chaotic attractors are recognizable in phase space. [C] The poincaré section shows chaos in the orbit, but that the motions are still constrained to a finite area in phase space.



[A] [B] [C]

Figure 10: [A] The observed motion when the driving arm is run with period $1.26 \pm 0.02s$ (near the resonance frequency) is chaotic. [B] . [C] The poincaré section shows the same integrated area of constraint in variation as the first test, but more evenly distributed.

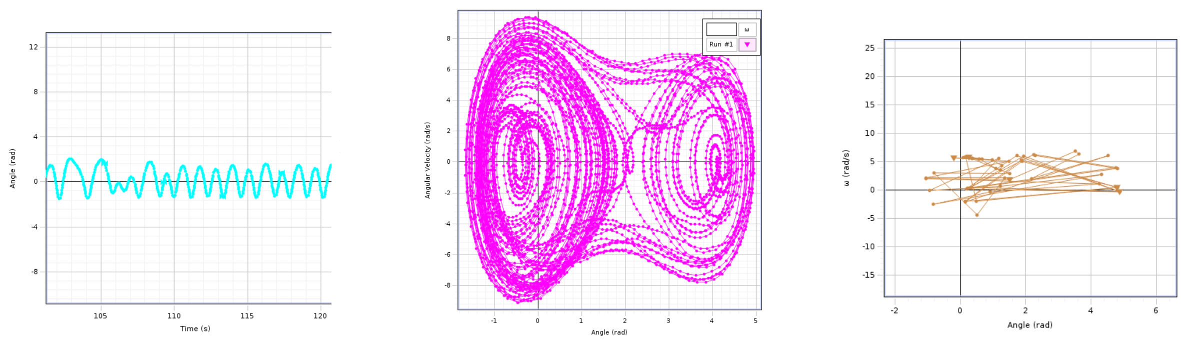


Figure 11: The observed motion when the driving arm is run with period $1.16 \pm 0.02s$. The time graph looks significiantly different from the previous tests, but the same attractors and poincaré section are observed.