For a particle in an infinite square well, calculate the probability of finding the particle in the range 3L/4 < x < L for each of the first three energy eigenstates.

The first three energy eigenstates are

$|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$.

The probability of finding a particle in a state n is

$$|\langle\psi_n|\psi_n\rangle|^2 = \left|\int_{-\infty}^{\infty} dx\ |\psi_n(x)|^2.\right|^2$$

But for just 3L/4 to L,

$$\langle\psi_n|\psi_n\rangle = \int_{3/4L}^{L} dx\ |\psi_n(x)|^2.$$

Since this is a particle in an infinite square well, the position space representation of an energy eigenstate n is

$$\psi_n(x) = \sqrt{\left(\frac{2}{L}\right)}\ \sin\left(\frac{n\pi x}{L}\right), \text{ so } |\psi_n(x)|^2 = \left(\frac{2}{L}\right)\ \sin^2\left(\frac{n\pi x}{L}\right).$$

Therefore, with the power reduction identity,

$$\langle\psi_n|\psi_n\rangle = \left(\frac{2}{L}\right) \int_{3/4L}^{L} dx\ \sin^2\left(\frac{n\pi x}{L}\right) = \left(\frac{2}{L}\right) \int_{3/4L}^{L} dx\ \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right)\right).$$

$$\langle\psi_n|\psi_n\rangle = \left(\frac{2}{L}\right)\ \left(\frac{1L}{8} - \left(\frac{1L}{4n\pi}\sin(2n\pi) - \frac{1L}{4n\pi}\sin\left(\frac{3n\pi}{2}\right)\right)\right)$$

$$= \frac{1}{16}\frac{L}{n\pi}\left(\sin(2n\pi) - \sin\left(\frac{3n\pi}{2}\right)\right).$$

In general, then, the probability of finding the particle in a particular energy state n is given by

$$|\langle\psi_n|\Psi\rangle|^2 = \left|\frac{1}{16}\frac{L}{n\pi}\left(\sin(2n\pi) - \sin\left(\frac{3n\pi}{2}\right)\right)\right|^2$$

$$= \frac{1}{256}\left(\frac{L^2}{n^2\pi^2}\right)\ \left|\left(\sin(2n\pi) - \sin\left(\frac{3n\pi}{2}\right)\right)\right|^2$$

$$= \frac{1}{256}\left(\frac{L^2}{n^2\pi^2}\right)\ \left(\sin(2n\pi) - \sin\left(\frac{3n\pi}{2}\right)\right)\ \left(\sin(2n\pi) - \sin\left(\frac{3n\pi}{2}\right)\right)$$

$$= \frac{1}{256}\left(\frac{L^2}{n^2\pi^2}\right)\ \left(\sin^2(2n\pi) + \sin^2\left(\frac{3n\pi}{2}\right) + 2\sin(2n\pi)\sin\left(\frac{3n\pi}{2}\right)\right).$$

The third term in the parentheses will always return 0, since one function or the other returns 0 depending on whether n is even or odd. The first term will also always be zero, since regardless of the value of n, the argument is an integer multiple of π. The final term will disappear when n is even, so the probability of finding a particle in the specified region for an energy state n is therefore

$$P(n) \begin{cases} 0, & n \text{ even} \\ \frac{1}{256}\left(\frac{L^2}{n^2\pi^2}\right)\ \left(\sin^2\left(\frac{3n\pi}{2}\right)\right), & n \text{ odd} \end{cases}$$

This is wrong, because for n=2, there should be some probability of finding the particle within this region, since the only node is at the center, and the wave function is non-zero across the region and does not cross the axis. This function will return a probability of 0 for n=2, which does not make sense with my statement. For n=1,3, at least a probability can be calculated:

$$P(n=1) = \frac{1}{256}\left(\frac{L^2}{\pi^2}\right)\sin^2\left(\frac{3\pi}{2}\right) = \frac{1}{256}\left(\frac{L^2}{\pi^2}\right) = 3.96*10^{-4}\ L^2$$

$$P(n=3) = \frac{1}{256}\left(\frac{L^2}{9\ \pi^2}\right)\left(\sin^2\left(\frac{9\pi}{2}\right)\right) = \frac{1}{2304}\left(\frac{L^2}{\pi^2}\right) = 4.40*10^{-5}\ L^2$$