Two operators' matrix representations are known in the $|1\rangle$, $|2\rangle$, $|3\rangle$ basis, where a and b are real numbers, and ι is the imaginary unit:

$$\hat{A} \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad \text{and} \quad \hat{B} \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\iota b \\ 0 & \iota b & 0 \end{pmatrix}.$$

B exhibits a degenerate spectrum when it has repeated eigenvalues. The eigenvalues of B are obtained from its characteristic equation.

$$\det\left(\begin{vmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -\iota b \\ 0 & \iota b & -\lambda \end{vmatrix} \right) = 0, \text{ i.e.}$$

$$(b - \lambda)(\lambda^2 + \iota^2 b^2) = (b - \lambda)(\lambda^2 - b^2) = (b - \lambda)(b - \lambda)(b + \lambda) = 0.$$

(**a**) The eigenvalues for this operator are therefore $\lambda = b,b,-b$. Since b appears twice, the operator exhibits a degenerate spectrum.

To find if A and B commute, their commutator need be evaluated. They commute if the value is 0. The commutator of two operators is defined as

$$[\hat{A},\hat{B}] = \hat{A}\,\hat{B} - \hat{B}\,\hat{A}.$$

For the given operators, then, the commutator is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\iota b \\ 0 & \iota b & 0 \end{pmatrix} - \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\iota b \\ 0 & \iota b & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & \iota ab \\ 0 & -\iota ab & 0 \end{pmatrix} - \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & \iota ab \\ 0 & -\iota ab & 0 \end{pmatrix} = 0.$$

(**b**) Therefore, these operators commute.

Since the operators commute, they share a set of common eigenstates. The eigenstates of $\hat{A}$ are apparent from inspection:

$$|a_1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |a_2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \text{and} \quad |a_3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Eigenstates of $\hat{B}$ must be obtained, and if they are a linear combination of the eigenstates of $\hat{A}$, they will serve as the complete set of eigenstates shared by $\hat{A}$ and $\hat{B}$. If not, some subset of those will be the shared basis, and this will need to be determined.

For $\hat{B}$, the eigenvalues are already known ($\lambda = b,b,-b.$), and using the eigenvalue equations, the eigenstates can be determined. The eigenvalue equation

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\iota b \\ 0 & \iota b & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} b\,\alpha \\ -\iota\, b\, \gamma \\ \iota\, b\, \beta \end{pmatrix} = b \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

dicates two possible eigenstates (for the eigenvalue b). One eigenstate is obvious from inspection:

$$|b_1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalue equation also reveals $-\iota\,\gamma = \beta$. If $\beta = 1$ is chosen, then $\gamma = -\iota$, revealing a second eigenstate, after normalizing:

$$|b_2\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -\iota \end{pmatrix}.$$

Similarly, when the eigenvalue -b is used, the eigenvalue equation reveals $\iota\,\gamma = \beta$. So, if $\beta = \iota$, $\gamma = 1$. The third eigenstate is therefore, after normalizing,

$$|b_3\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \iota \\ 1 \end{pmatrix}$$

The complete set of eigenstates of the operator $\hat{B}$ is

$$|b_1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |b_2\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -\iota \end{pmatrix}, \text{ and } \quad |b_3\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \iota \\ 1 \end{pmatrix}.$$

(**c**) These eigenstates can be expressed as linear combinations of the eigenstates of $\hat{A}$, so they are shared eigenstates between these operators, and so this basis is simultaneously a basis of $\hat{A}$ and $\hat{B}$.