If S_y and S_x are spin-1 operators in the z basis, what are the results if $S_y(S_y + \hbar)(S_y - \hbar)$ and $S_x(S_x + \hbar)(S_x - \hbar)$ are evaluated?

The spin eigenstates and eigenvalues are known from experiment for a spin-1 system, and the spin-z and spin-x operators in the z basis, S_y and S_x, have the following matrix representations:

$$S_y \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and}$$

$$S_x \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Using the matrix representations, the expressions can be evaluated. For the spin-z operator, the expression $S_y(S_y + \hbar)(S_y - \hbar) \doteq$

$$\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \left[\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \left[\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right],$$

which simplifies to the matrix multiplication operation, where $\mathbb{0}$ represents the 0 matrix,

$$\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \hbar \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{0}.$$

The multiplication operation apparently returns $\mathbb{0}$ because the third factor will nullify any terms besides center terms, and the first factor will nullify any center terms.

Similarly, $S_x(S_x + \hbar)(S_x - \hbar) \doteq$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}.$$

Performing the multiplication operation on the last two matrices returns the expression

$$\frac{\hbar^3}{2\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \mathbb{0}.$$

Because each pair of row and column in this matrix has alternating 0s and ±1s, every multiplication operation will return 0. The second expression is therefore equivalent to the zero matrix $\mathbb{0}$.