The harmonic oscillator potential is $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$. The hamiltonian is time independent.

The initial state vector is

$$\Psi(t=0) = A\left(|0\rangle + 2\exp\left(\frac{\imath\pi}{2}\right)|1\rangle\right)$$

Finding the normalization constant A is simple enough, since

$$1 = A^2 + \left(2A\exp\left(\frac{\imath\pi}{2}\right)\right)^2 = A^2\left(1 + \left(2\exp\left(\frac{\imath\pi}{2}\right)\right)^2\right).$$

$$A = \sqrt{\left(\frac{1}{1 + \left(2\exp\left(\frac{\imath\pi}{2}\right)\right)^2}\right)} = \sqrt{\left(\frac{1}{1 + 4\exp\left(\frac{\imath\pi}{2} - \frac{\imath\pi}{2}\right)}\right)},$$

(**a**)

$$A = \sqrt{\left(\frac{1}{1 + 4\exp(0)}\right)} = \sqrt{\frac{1}{5}}.$$

The time evolution prescription for a time-independent hamiltonian is

$$\Psi(t) = \exp\left(-\frac{\imath E_n t}{\hbar}\right)\Psi(t=0);$$

$$\Psi(t) = \exp\left(-\frac{\imath E_n t}{\hbar}\right)\sqrt{\frac{1}{5}}\left(|0\rangle + 2\exp\left(\frac{\imath\pi}{2}\right)|1\rangle\right).$$

(**b**)

$$\Psi(t) = \sqrt{\frac{1}{5}}\left(\exp\left(-\frac{\imath E_0 t}{\hbar}\right)|0\rangle + 2\exp\left(\imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)|1\rangle\right).$$

The expecation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ are of interest.

$$\langle\hat{x}\rangle = \langle\hat{\Psi}(t)|\hat{x}|\Psi(t)\rangle.$$

$$\hat{\Psi}(t) = \sqrt{\frac{1}{5}}\left(\exp\left(\frac{\imath E_0 t}{\hbar}\right)\langle 0| + 2\exp\left(\imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\langle 1|\right).$$

$$\langle\hat{x}\rangle = \frac{1}{5}\left(\exp\left(\frac{\imath E_0 t}{\hbar}\right)\langle 0| + 2\exp\left(\imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\langle 1|\right)\hat{x}\left(\exp\left(-\frac{\imath E_0 t}{\hbar}\right)|0\rangle + 2\exp\left(\imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)|1\rangle\right).$$

The matrix representations of the position operator $\hat{x}$ and momentum operator $\hat{p}$ have been developed from the definition of the increment/decrement operators. The matrix elements may be ascertained by inspection.

$$\hat{x} \doteq \begin{pmatrix} 0 & \sqrt{1} \\ \sqrt{1} & 0 \end{pmatrix}, \qquad \hat{p} \doteq \begin{pmatrix} 0 & -\imath\sqrt{1} \\ \imath\sqrt{1} & 0 \end{pmatrix}.$$

$$\langle 0|\hat{x}|0\rangle = x_{00} = \langle 1|\hat{x}|1\rangle = x_{11} = 0.$$

$$\langle 0|\hat{x}|1\rangle = x_{01} = \langle 1|\hat{x}|0\rangle = x_{10} = \sqrt{1} = 1.$$

$$\langle\hat{x}\rangle = \frac{1}{5}\left(\begin{array}{l} \exp\left(\frac{\imath E_0 t}{\hbar}\right)\exp\left(-\frac{\imath E_0 t}{\hbar}\right)\langle 0|\hat{x}|0\rangle + \\ \exp\left(\frac{\imath E_0 t}{\hbar}\right)2\exp\left(\imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)\langle 0|\hat{x}|1\rangle + \\ \exp\left(-\frac{\imath E_0 t}{\hbar}\right)2\exp\left(\imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\langle 1|\hat{x}|0\rangle + \\ 2\exp\left(\imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)2\exp\left(\imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)\langle 1|\hat{x}|1\rangle \end{array}\right).$$

Substituting the matrix elements:

$$\langle\hat{x}\rangle = \frac{2}{5}\left(\begin{array}{l} \exp\left(\frac{\imath E_0 t}{\hbar} + \imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right) + \\ \exp\left(-\frac{\imath E_0 t}{\hbar} + \imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right) \end{array}\right) = \frac{2}{5}\left(\begin{array}{l} \exp\left(\imath\left(\left(\frac{E_0 - E_1}{\hbar}\right)t + \frac{\pi}{2}\right)\right) + \\ \exp\left(\imath\left(\left(\frac{E_1 - E_0}{\hbar}\right) - \frac{\pi}{2}\right)\right) \end{array}\right);$$

$$\langle\hat{x}\rangle = \frac{2}{5}\left(\exp\left(\frac{\imath E_0 t}{\hbar} + \imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right) + \exp\left(-\frac{\imath E_0 t}{\hbar} + \imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\right);$$

$$\langle\hat{x}\rangle = \frac{2}{5}\left(\exp\left(\imath\left(\left(\frac{E_0 - E_1}{\hbar}\right)t + \frac{\pi}{2}\right)\right) + \exp\left(\imath\left(\left(\frac{E_1 - E_0}{\hbar}\right)t - \frac{\pi}{2}\right)\right)\right).$$

For the Harmonic Oscillator, $E_n = \hbar\omega(n + \frac{1}{2})$.

$E_n - E_{n'} = \hbar\omega(n - n')$:
$E_0 - E_1 = \hbar\omega(0 - 1) = -\hbar\omega$;
$E_1 - E_0 = \hbar\omega$.

$$\langle\hat{x}\rangle = \frac{2}{5}\left(\exp\left(\imath\left(\left(\frac{-\hbar\omega}{\hbar}\right)t + \frac{\pi}{2}\right)\right) + \exp\left(\imath\left(\left(\frac{\hbar\omega}{\hbar}\right)t - \frac{\pi}{2}\right)\right)\right).$$

$$\langle \hat{x} \rangle = \frac{1}{2}\left(e^{\imath\left(-\omega t + \frac{\pi}{2}\right)} + \exp\left(\imath\left(\omega t - \frac{\pi}{2}\right)\right) \right).$$

(**c**)
Then,

$$\langle \hat{x} \rangle = 0.$$