The harmonic oscillator potential is $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$. The hamiltonian is time independent.

The initial state vector is

$$\Psi(t=0) = A\left(|0\rangle + 2\exp\left(\frac{\imath\pi}{2}\right)|1\rangle\right)$$

Finding the normalization constant A is simple enough, since

$$1 = A^2 + \left(2A\exp\left(\frac{\imath\pi}{2}\right)\right)^2 = A^2\left(1 + \left(2\exp\left(\frac{\imath\pi}{2}\right)\right)^2\right).$$

$$A = \sqrt{\left(\frac{1}{1 + \left(2\exp\left(\frac{\imath\pi}{2}\right)\right)^2}\right)} = \sqrt{\left(\frac{1}{1 + 4\exp\left(\frac{\imath\pi}{2} - \frac{\imath\pi}{2}\right)}\right)},$$

(**a**)

$$A = \sqrt{\left(\frac{1}{1 + 4\exp(0)}\right)} = \sqrt{\frac{1}{5}}.$$

The time evolution prescription for a time-independent hamiltonian is

$$\Psi(t) = \exp\left(-\frac{\imath E_n t}{\hbar}\right)\Psi(t=0);$$

$$\Psi(t) = \exp\left(-\frac{\imath E_n t}{\hbar}\right)\sqrt{\frac{1}{5}}\left(|0\rangle + 2\exp\left(\frac{\imath\pi}{2}\right)|1\rangle\right).$$

(**b**)

$$\Psi(t) = \sqrt{\frac{1}{5}}\left(\exp\left(-\frac{\imath E_0 t}{\hbar}\right)|0\rangle + 2\exp\left(\imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)|1\rangle\right).$$

The expecation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ are of interest.

$$\langle\hat{x}\rangle = \langle\Psi(t)|\hat{x}|\Psi(t)\rangle.$$

$$\Psi(t) = \sqrt{\frac{1}{5}}\left(\exp\left(\frac{\imath E_0 t}{\hbar}\right)\langle 0| + 2\exp\left(\imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\langle 1|\right).$$

$$\langle\hat{x}\rangle = \frac{1}{5}\left(\exp\left(\frac{\imath E_0 t}{\hbar}\right)\langle 0| + 2\exp\left(\imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\langle 1|\right)\hat{x}\left(\exp\left(-\frac{\imath E_0 t}{\hbar}\right)|0\rangle + 2\exp\left(\imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)|1\rangle\right).$$

The matrix representations of the position operator $\hat{x}$ and momentum operator $\hat{p}$ have been developed from the definition of the increment/decrement operators. The matrix elements may be ascertained by inspection.

$$\hat{x} \doteq \sqrt{\left(\frac{\hbar}{2m\omega}\right)}\begin{pmatrix} 0 & \sqrt{1} \\ \sqrt{1} & 0 \end{pmatrix}, \quad \hat{p} \doteq \sqrt{\left(\frac{\hbar m\omega}{2}\right)}\begin{pmatrix} 0 & -\imath\sqrt{1} \\ \imath\sqrt{1} & 0 \end{pmatrix}.$$

ω is a characteristic parameter of the system. It is related to the steepness of the parabolic potential curve. m is the particle's mass.

$$\langle\hat{x}\rangle = \frac{1}{5}\left(\begin{array}{l} \exp\left(\frac{\imath E_0 t}{\hbar}\right)\exp\left(-\frac{\imath E_0 t}{\hbar}\right)\langle 0|\hat{x}|0\rangle + \\ \exp\left(\frac{\imath E_0 t}{\hbar}\right)2\exp\left(\imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)\langle 0|\hat{x}|1\rangle + \\ \exp\left(-\frac{\imath E_0 t}{\hbar}\right)2\exp\left(\imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\langle 1|\hat{x}|0\rangle + \\ 2\exp\left(\imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)2\exp\left(\imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)\langle 1|\hat{x}|1\rangle \end{array}\right).$$

$$\langle 0|\hat{x}|0\rangle = x_{00} = \langle 1|\hat{x}|1\rangle = x_{11} = 0.$$

$$\langle 0|\hat{x}|1\rangle = x_{01} = \langle 1|\hat{x}|0\rangle = x_{10} = \sqrt{\left(\frac{\hbar}{2m\omega}\right)}.$$

Substituting the matrix elements:

$$\langle\hat{x}\rangle = \frac{1}{5}\sqrt{\frac{2\hbar}{m\omega}}\left(\exp\left(\frac{\imath E_0 t}{\hbar} + \imath\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right) + \exp\left(-\frac{\imath E_0 t}{\hbar} + \imath\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\right);$$

$$\langle\hat{x}\rangle = \frac{1}{5}\sqrt{\frac{2\hbar}{m\omega}}\left(\exp\left(\imath\left(\left(\frac{E_0 - E_1}{\hbar}\right)t + \frac{\pi}{2}\right)\right) + \exp\left(\imath\left(\left(\frac{E_1 - E_0}{\hbar}\right)t - \frac{\pi}{2}\right)\right)\right).$$

For the Harmonic Oscillator, $E_n = \hbar\omega(n + \frac{1}{2})$.

$E_n - E_{n'} = \hbar\omega(n - n')$:

$E_0 - E_1 = \hbar\omega(0 - 1) = -\hbar\omega$;

$E_1 - E_0 = \hbar\omega$.

$$\langle \hat{x} \rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\exp\left(\iota\left(\left(\frac{-\hbar\omega}{\hbar}\right)t + \frac{\pi}{2}\right)\right) + \exp\left(\iota\left(\left(\frac{\hbar\omega}{\hbar}\right)t - \frac{\pi}{2}\right)\right) \right).$$

$$\langle \hat{x} \rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\ \frac{\exp(\iota\pi)}{2}\left(\exp\left(\iota\left(\frac{-\hbar\omega}{\hbar}\right)t\right) + \exp\left(\iota\left(\left(\frac{\hbar\omega}{\hbar}\right)t - \pi\right)\right) \right).$$

$$\langle \hat{x} \rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\ \frac{\exp(\iota\pi)}{2}\left(\exp\left(\iota\left(\frac{-\hbar\omega}{\hbar}\right)t\right) - \exp\left(\iota\left(\frac{\hbar\omega}{\hbar}\right)t\right) \right).$$

This is a sine function.

$$\langle \hat{x} \rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\ \frac{\exp(\iota\pi)}{2}\ 2\iota\ \sin\left(\frac{-\hbar\omega t}{\hbar}\right) = \frac{1}{2}\sqrt{\frac{8\hbar}{m\omega}}\ \iota^2\ -\sin(\omega t).$$

(**c**, $\hat{x}$)

$$\langle \hat{x} \rangle = \frac{1}{2}\sqrt{\frac{8\hbar}{m\omega}}\ \sin(\omega t).$$

The expectation value progresses periodically with the time parameter t.

A very similar argument can be made for the momentum operator.

$$\langle \hat{p} \rangle = \frac{1}{2}\left(\begin{array}{l} \exp\left(\frac{\iota E_0 t}{\hbar}\right)\ \exp\left(\frac{-\iota E_0 t}{\hbar}\right)\langle 0|\hat{p}|0\rangle + \\ \exp\left(\frac{\iota E_0 t}{\hbar}\right)\ 2\ \exp\left(\iota\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)\ \langle 0|\hat{p}|1\rangle + \\ \exp\left(\frac{-\iota E_0 t}{\hbar}\right)\ 2\ \exp\left(\iota\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\ \langle 1|\hat{p}|0\rangle + \\ 2\ \exp\left(\iota\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right) 2\ \exp\left(\iota\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right)\langle 1|\hat{p}|1\rangle \end{array}\right).$$

$$\langle 0|\hat{p}|0\rangle = p_{00} = \langle 1|\hat{p}|1\rangle = p_{11} = 0.$$

$$\langle 0|\hat{p}|1\rangle = p_{01} = -\iota\sqrt{\left(\frac{\hbar m\omega}{2}\right)}.$$

$$\langle 1|\hat{p}|0\rangle = p_{10} = \iota\sqrt{\left(\frac{\hbar m\omega}{2}\right)}.$$

$$\langle \hat{p} \rangle = \frac{2}{2}\ \iota\ \sqrt{\left(\frac{\hbar m\omega}{2}\right)}\left(-\exp\left(\frac{\iota E_0 t}{\hbar} + \iota\left(\frac{\pi}{2} - \frac{E_1 t}{\hbar}\right)\right) + \exp\left(-\frac{\iota E_0 t}{\hbar} + \iota\left(\frac{E_1 t}{\hbar} - \frac{\pi}{2}\right)\right)\right).$$

$$\langle \hat{p} \rangle = \frac{2}{2}\ \iota\ \sqrt{\left(\frac{\hbar m\omega}{2}\right)}\left(\exp\left(\iota\left(\left(\frac{-\hbar\omega}{\hbar}\right)t - \frac{\pi}{2}\right)\right) + \exp\left(\iota\left(\left(\frac{\hbar\omega}{\hbar}\right)t - \frac{\pi}{2}\right)\right)\right).$$

$$\langle \hat{p} \rangle = \frac{2}{2}\ \iota\ \sqrt{\left(\frac{\hbar m\omega}{2}\right)}\ \exp\left(\frac{-\iota\pi}{2}\right)\ \left(\exp\left(\iota\left(\frac{-\hbar\omega}{\hbar}\right)t\right) + \exp\left(\iota\left(\frac{\hbar\omega}{\hbar}\right)t\right) \right).$$

This is a cosine.

(**c**, $\hat{p}$)

$$\langle \hat{p} \rangle = \frac{1}{4}\ \iota\ \sqrt{\left(\frac{\hbar m\omega}{2}\right)}\ \exp\left(\frac{-\iota\pi}{2}\right)\ \cos(\omega t) = \frac{1}{2}\sqrt{(8\hbar m\omega)}\ \cos(\omega t).$$

Ehrenfest's theorem states

$$\langle \hat{p} \rangle = m\ \frac{d\langle \hat{x} \rangle}{dt}.$$

$$m\ \frac{d\langle \hat{x} \rangle}{dt} = \frac{1}{2}\ m\omega\ \sqrt{\frac{8\hbar}{m\omega}}\ \cos(\omega t) = \frac{1}{2}\sqrt{(8m\omega\hbar)}\ \cos(\omega t) = \langle \hat{p} \rangle.$$

So, the theorem holds for this case.