$|a\rangle$ and $|b\rangle$ are eigenstates of a Hermitian operator A with eigenvalues a and b, $a \neq b$. The Hamiltonian operator is

$\hat{H} = |a\rangle\ \delta\ \langle a| + |b\rangle\ \delta\ \langle b|$, with δ a real number.

a) The eigenstates of the Hamiltonian can be determined by diagonalizing the Hamiltonian operator's matrix representation. In general,

$$\hat{H} \doteq \begin{pmatrix} \langle a|\hat{H}|a\rangle & \langle a|\hat{H}|b\rangle \\ \langle b|\hat{H}|a\rangle & \langle b|\hat{H}|b\rangle \end{pmatrix}.$$

Calculating the individual components:

$$\langle a|\hat{H}|a\rangle = \langle a|(|a\rangle\ \delta\ \langle a| + |b\rangle\ \delta\ \langle b|)|a\rangle =$$

$$\langle a|\hat{H}|a\rangle = \langle a|a\rangle\ \delta\ \langle a|a\rangle + \langle a|b\rangle\ \delta\ \langle b|a\rangle =$$

$$\langle a|\hat{H}|a\rangle = \delta(1 + \langle a|b\rangle\langle b|a\rangle),$$

and because H is a hermitian operator, $\langle a|b\rangle = \langle b|a\rangle$, so

$$\langle a|\hat{H}|a\rangle = \delta(1 + |\langle a|b\rangle|^2);$$

$$\langle a|\hat{H}|b\rangle = \langle a|a\rangle\ \delta\ \langle a|b\rangle + \langle a|b\rangle\ \delta\ \langle b|b\rangle =$$

$$\langle a|\hat{H}|b\rangle = \delta\ (\langle a|b\rangle + \langle a|b\rangle) = \delta\ 2\langle a|b\rangle;$$

because of the hermition property,
$\langle b|\hat{H}|a\rangle = \langle a|\hat{H}|b\rangle = \delta\ 2\langle a|b\rangle;$

finally,

$$\langle b|\hat{H}|b\rangle = \langle b|(|a\rangle\ \delta\ \langle a| + |b\rangle\ \delta\ \langle b|)|b\rangle =$$

$$\langle b|\hat{H}|b\rangle = \langle b|a\rangle\ \delta\ \langle a|b\rangle + \langle b|b\rangle\ \delta\ \langle b|b\rangle =$$

$$\langle b|\hat{H}|b\rangle = \delta(\langle b|a\rangle\langle a|b\rangle + 1),$$

$$\langle b|\hat{H}|b\rangle = \delta(1 + |\langle a|b\rangle|^2).$$

So, the Hamiltonian operator $\hat{H} \doteq$

$$\delta \begin{pmatrix} 1 + |\langle a|b\rangle|^2 & 2\langle a|b\rangle \\ 2\langle a|b\rangle & 1 + |\langle a|b\rangle|^2 \end{pmatrix}.$$

The eigenstates, which I will call $|1\rangle$ and $|2\rangle$ can be obtained by diagonalizing the Hamiltonian matrix. The first eigenvalue equations are

$\hat{H}|1\rangle = E_1|1\rangle$ and $\hat{H}|2\rangle = E_2|2\rangle$, with the eigenstates represented by the vector matrices, respectively,

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}.$$

$$\delta \begin{pmatrix} 1 + |\langle a|b\rangle|^2 & 2\langle a|b\rangle \\ 2\langle a|b\rangle & 1 + |\langle a|b\rangle|^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = E_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}.$$

This gives the equation $\alpha_1 + \alpha_1|\langle a|b\rangle|^2 + 2\beta_1\langle a|b\rangle = E_1\alpha_1$, and therefore the ratio between α_1 and β_1,

$$\frac{\beta_1}{\alpha_1} = \frac{E_1 - 1 - |\langle a|b\rangle|^2}{2\langle a|b\rangle}, \text{ or}$$

$$\beta_1 = \frac{\alpha_1(E_1 - 1 - |\langle a|b\rangle|^2)}{2\langle a|b\rangle}$$

Using the normalization condition, the values of each constant can be obtained. Plugging the value for α_1 into the equation reveals a quadratic equation.

$|\alpha_1|^2 + |\beta_1|^2 = 1$, so

$$|\alpha_1|^2 + \left| \frac{\alpha_1(E_1 - 1 - |\langle a|b\rangle|^2)}{2\langle a|b\rangle} \right|^2 = 1.$$

$\alpha_1(1 - E_1) + 2\beta_1\langle a|b\rangle + \alpha_1|\langle a|b\rangle|^2 = 0$ and

$\alpha_1 = \pm\sqrt{1 - |\beta_1|^2}$, so

$\pm\sqrt{1 - |\beta_1|^2}\,(1 - E_1) + 2\beta_1\langle a|b\rangle + \pm\sqrt{1 - |\beta_1|^2}\,|\langle a|b\rangle|^2 = 0.$

The quadratic formula therefore says that