Two operators' matrix representations are known in the |1⟩, |2⟩, |3⟩ basis, where and b are real numbers:

$$A \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad \text{and} \quad B \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\imath b \\ 0 & \imath b & 0 \end{pmatrix}.$$

B exhibits a degenerate spectrum when it has repeated eigenvalues. The eigenvalues of B are obtained from its characteristic equation.

$$\left| \begin{pmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -\imath b \\ 0 & \imath b & -\lambda \end{pmatrix} \right| = 0, \text{ i.e.}$$

$$(b - \lambda)(\lambda^2 + \imath^2 b^2) = (b - \lambda)(\lambda^2 - b^2) = (b - \lambda)(b - \lambda)(b + \lambda) = 0.$$

(a) The eigenvalues for this operator are therefore λ = b,b,-b. Since b appears twice, the operator exhibits a degenerate spectrum.

To find if A and B commute, their commutator need be evaluated. They commute if the value is 0. The commutator of two operators is defined as

$$[\hat{A},\hat{B}] = \hat{A}\,\hat{B} - \hat{B}\,\hat{A}.$$

For the given operators, then, the commutator is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\imath b \\ 0 & \imath b & 0 \end{pmatrix} - \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\imath b \\ 0 & \imath b & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & \imath ab \\ 0 & -\imath ab & 0 \end{pmatrix} - \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & \imath ab \\ 0 & -\imath ab & 0 \end{pmatrix} = 0.$$

(b) Therefore, these operators commute.

Since the operators commute, they share common eigenstates. Therefore, if the eigenstates for one operator can be determined, they are determined for both operators. Since the problem states that a new set of orthonormal kets need be determined, and the given set are the eigenstates related to the operator A, then the eigenstates of the operator B should be determined.

The eigenvalues are already known (λ = b,b,-b.), and using the eigenvalue equations, the eigenstates can be determined. The eigenvalue equation

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\imath b \\ 0 & \imath b & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} b\,\alpha_1 \\ -\imath\, b\, \gamma_1 \\ \imath\, b\, \beta_1 \end{pmatrix} = b \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \text{ reveals}$$

$$-\imath\, \gamma_1 = \beta_1, \text{ and}$$
$$\imath\, b\, \beta_1 = \gamma_1$$