A spin-1/2 particle has a magnetic moment **μ** and is placed in a uniform magnetic field **B**, which is aligned with the z axis, so $\mathbf{B} = B_z \hat{z}$. It is known that the Hamiltonian operator for this system commutes with the spin component operator in the z direction but not with spin component operators in the x and y directions. How and why is this important?

The commutator is defined as $[\hat{u},\hat{v}] = \hat{u}\hat{v}-\hat{v}\hat{u}$, where $\hat{u}$ and $\hat{v}$ are operators.

If the product of $\hat{u}$ and $\hat{v}$ is independent of order, the commutator will be zero, and the operators can be said to commute. To test whether the Hamiltonian and the spin operators commute, one need express them in a common basis and compute the commutator. The z basis should serve as a natural basis for the comparison.

The spin operator in the z basis $\hat{S}_z$ is diagonalized in the z basis because the magnetic field is oriented in the z direction. This result is known from the Stern-Gerlach experiment.

$$\hat{S}_z \doteq \hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The Hamiltonian H is the total energy of the system. The only contributing term for this system is the potential energy $V = -\boldsymbol{\mu}\cdot\mathbf{B}$, where **μ** and **B** are the physical terms described in the introduction. The magnetic moment of a magnetic dipole, such as that seen with a spin-1/2 particle, is $\boldsymbol{\mu} = g\, q/2m_e\, \mathbf{S}$, with **S** the intrinsic spin vector for the case of no orbital angular momentum, and the remaining factors constant, intrinsic values of the particle. Then, with

$$k \stackrel{\text{def}}{=} g\, q/2m_e,\ \boldsymbol{\mu} = k\, \mathbf{S}.$$

The Hamiltonian H is therefore, because $\mathbf{B} = B_z \hat{z}$,

$$H = V = -\boldsymbol{\mu}\cdot\mathbf{B} = -k\, S_z B_z.$$

The Hamiltonian is measureable, and therefore is an operator. The magnetic field strength is constant in this system, so the Hamiltonian operator is related to the spin operator in the z basis by

$$\hat{H} = -k\, B_z \hat{S}_z.$$

Therefore, the Hamiltonian

$$\hat{H} \doteq -k\, B_z \hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The commutator can now be computed. A constant factor of $-k\, B_z \hbar^2/4$ can be collected, leaving only the matrix multiplication to determine commutability.

$$[\hat{H},\hat{S}_z] = \hat{H}\,\hat{S}_z - \hat{S}_z\hat{H}.$$

$$K \stackrel{\text{def}}{=} -k\, B_z \hbar^2/4.$$

$$\hat{H}\,\hat{S}_z \doteq K \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\hat{S}_z\hat{H} \doteq K \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The computation indicates that $\hat{S}_z\hat{H} = \hat{H}\,\hat{S}_z$, and therefore the commutator is zero. The Hamiltonian and the spin operator in the z direction commute. A similar computation for the x and y directions should indicate a lack of commutability.

The spin operators in the x direction and z direction in the z basis, with $\iota$ the imaginary unit,

$$\hat{S}_x \doteq \hbar/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\hat{S}_y \doteq \hbar/2 \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}.$$

As with the spin component in the z direction, when multiplying the Hamiltonian and spin in x or spin in y operators, the multiplicative factors associate outside the matrices, and only the matrix multiplication determines the commutability of the operators, so, the operators are commutable if their matrix components are commutable.

For $\hat{H}$ and $\hat{S}_x$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \text{ and}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since these matrix products are not equal, their difference is not zero. Therefore, these operators do not commute.

For $\hat{H}$ and $\hat{S}_y$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\iota \\ -\iota & 0 \end{pmatrix}; \text{ and}$$

$$\begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}.$$

Again, these matrix products are not equal, so these operators do not commute.

When an observable, such as the spin component operator in the z direction $\hat{S}_z$, commutes with the Hamiltonian, it is referred to as a constant of motion. Such a quantity has specific values its measurement will consistently result in. Since the Hamiltonian does not commute with the remaining spin components, their values may vary.